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Quantum Statistical Mechanics for Superstable Interactions: Bose–Einstein Statistics

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We study quantum systems of interacting Bose particles confined to a bounded region Λ of $\mathbf{R}^\nu$. For any superstable and (strong) lower regular interaction, we obtain uniform bounds on the expectations of exponentials of local number operators for any activity and for any temperature. The method we use here is an improvement over our previous method on the same subject. As a consequence of the bounds, any infinite volume limit states are entire analytic and locally normal. Furthermore under an integrability condition on the interaction, the limit states are modular states. In this case, we use the Green's function method to construct an infinite volume limit Hilbert spece, a strongly continuous time evolution group of unitary operators and an invariant vector. Moreover we prove the existence of the pressure and its independence of boundary conditions.

KEY WORDS: Superstable interaction; Bose–Einstein statistics; local number operators; locally normal states; modular states; Wiener integral formalism; pressure; boundary conditions.

1. INTRODUCTION

In this paper we continue our study, initiated in Ref. 10, of statistical systems of quantum particles obeying Bose–Einstein statistics and interacting via a superstable and lower regular potential. For the classical systems with superstable interactions, Ruelle established uniform bounds on the finite volume correlation functions.[14] Using the bounds he obtained various results on the infinite volume equilibrium states and the pressure. In Ref. 10 we have extended *partially* Ruelle's results to the quantum systems of Bose particles. Because of technical difficulties we have assumed a condition stronger than superstability on the interactions,

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namely, the strong superstability. Although the results can be applied to physically interesting various interactions such as Lennard-Jones types,[10] the method was not suitable to deal with all superstable interactions. Recently we came to know that Esposito, Nicolo, and Pulvirenti have extended the classical results to the quantum systems obeying Maxwell–Boltzmann statistics.[2] In this paper we modify and improve the method we developed in Ref. 10 to include the basic ideas in Ref. 2 so that the main results in Ref. 10 hold for all superstable interactions. Thus the result of this paper is the answer to the problem (c) we proposed in the Introduction of Ref. 10. The improved method also enables us to prove the existence of the pressure and its independence of boundary conditions.

It may be worthwhile to remark on some results in quantum statistical mechanics. There have been extensive studies on the thermodynamic limit in statistical mechanics of interacting quantum systems, and fairly satisfactory results have been obtained for the thermodynamic functions such as pressure.[1,9,13] On the other hand, the results concerning the equilibrium states for such systems are less satisfactory. In the dilute regime, detailed properties of the infinite volume (thermodynamic) limit states have been obtained for various classes of interactions.[1,3,4,6,16] The charge conjugate invariant systems have also been studied in Ref. 5. It has to be pointed out that our approach is not perturbative, so it works at all values of temperatures and activities. In passing we mention that Ruelle's extimates[14] have been extended to unbounded classical and quantum spin systems.[15,8,11]

We present a brief discussion on the main results of this paper below. Let ρ_Λ be the finite volume Gibbs state for a system of interacting Bose particles confined to a bounded region Λ of the configuration space $\mathbf{R}^\nu$ and let N_B be the local number operator for $B \subset \Lambda$. We assume that the interaction satisfies the superstability and (strong) lower regularity conditions (see Section 2 for the definitions). We then obtain the bounds of the form ($\nu < 4$ if the potential is not repulsive):

$$\rho_\Lambda(\exp[aN_B]) \leqslant \exp[A(B, a)]$$

where $A(B, a)$ is a constant depending only on the diameter of B and $|a|$. We will give the exact definitions of the models, conditions on the interactions, and the main results in Section 2. As a consequence of the bounds it follows that any limit point ρ of the state ρ_Λ in the weak* topology is entirely analytic and locally normal in the CCR algebra. Furthermore under an additional assumption on the integrability of the potential, the state ρ is a modular state. In this case we use the Green's function method[1] to construct an infinite volume physical Hilbert space, a strongly continuous time evolution group of unitary operators and invariant vector(s). We also show the existence of the pressure and its independence of boundary conditions.

We next give a breif description of the basic ingredients of the method we use in this paper. The general strategy is essentially the same as that of Ref. 10. As in Ref. 10 we will use the Ginibre's representation of the partition function in terms of the Wiener integrals,[6,1,5,7] and a modification of Ruelle's method used for classical systems.[14] For quantum systems one has to control (a) quantum statistics and (b) interactions between two group of Wiener paths (W terms). Compared to the classical systems, these problems originate from the fluctuations (deviations) of Wiener paths. As in Ref. 10 we will use the fact that the Wiener measure of a subset of the paths with large deviations is small. On the other hand, the system behaves like a classical one on a subset with small deviations. The basic idea is to decompose the Wiener space Ω^n of n paths into disjoint subsets according to amount of fluctuations. On each subset we utilize the above facts to obtain uniform bounds. The method we developed in Ref. 10 is strong enough to control quantum statistics (B-E statistics), but not strong enough to control W terms for all superstable interactions. This was the reason why we assumed a stronger condition than superstability. On the other hand the method developed in Ref. 2 is sufficiently good to handle interaction terms for all superstable interactions, yet the method cannot be applied directly to solve quantum statistics. Thus we combine and improve two methods to control (a) and (b) for any superstable interactions.

We are unable to obtain pointwise bounds on reduced density matrices. But we believe the method we used can be extended to give such bounds. By establishing decay properties one ought to be able to see whether or not the systems do exhibit the Bose–Einstein condensations. As mentioned in Ref. 10, the Wiener integral formalism does seem to be unsuitable to handle the systems of interacting Fermi particles. Thus it should be desirable to develop a method in the pure operator approach, i.e., the second quantization formalism. That kind of a method should be useful to study composite systems such as matter systems.

We try to make this paper as self-contained as possible. Therefore, as we follow the general procedures of Ref. 10, it is necessary both to introduce some new notations and to repeat some arguments from Ref. 10. To make the paper a reasonable size we freely use some technical results of Ref. 10 without the proofs whenever its proofs do not affect main arguments of the paper.

The contents of the paper are as follows: In Section 2.1 We introduce notations and the definitions of the models. In Section 2.2 we list the assumption on the interactions (Assumption A) and then give our main estimate (Proposition 2.2.1). Then, the entire analyticity and local normality of any infinite volume limit state follow from the main estimate and the arguments used in Refs. 10 and 1. Under an assumption of the interactions (an integrability assumption) we use the main estimate and the Green's function method to construct an infinite volume limit theory (Theorem 2.2.3).

Sections 3 and 4 are devoted to the proof of the main estimate. In Section 3.1 we review the Wiener integral formalism in statistical mechanics and we then introduce a decomposition of the space of Wiener paths into multually disjoint subsets. With an assumption of one estimate (Theorem 3.1.1) we control quantum statistics in Section 3.2. In Section 4, we prove Theorem 3.1.1 and so we complete the proof of our main estimate. In Section 5 we finally show the existence of the pressure in the infinite volume limit and its independence of boundary conditions.

Before closing the Introduction we would like to make an apology to the readers for some printing (and typing) errors in Ref. 10. Although the correct meanings are obvious if one reads carefully, we make corrections of those errors in the Appendix of this paper.

2. THE DEFINITION OF MODELS AND THE MAIN RESULTS

2.1. Some Notations and Definitions

We first introduce the Hilbert space and the finite volume Gibbs states for a system of interacting Bose particles confined to a bounded open region Λ of the configuration space $\mathbf{R}^\nu$. Let

$$\mathscr{H}_n^{(s)}(\Lambda) = \bigotimes_{i=1}^{n}{}_s\, L^2(\Lambda, d^\nu x_i) \tag{2.1.1}$$

be the subspace of $L^2(\Lambda^n, d^{n\nu}x)$ formed by the totally symmetric functions of n variables $x_i \in \Lambda$. The associated Fock space

$$\mathsf{F}^{(s)}(\Lambda) = \bigoplus_{n=0} \mathscr{H}_n^{(s)}(\Lambda) \tag{2.1.2}$$

describe the states of an arbitrary number of particles. The total Hamiltonian is given by

$$H_\Lambda = \bigoplus_{n \geqslant 0} H_\Lambda^{(n)} \tag{2.1.3}$$

in terms of the n-particle Hamiltonian $H_\Lambda^{(n)}$ which has the form

$$H_\Lambda^{(n)} = -\frac{1}{2} \sum_{i=1}^{n} \Delta_{\Lambda,i} + U((x)_n) \tag{2.1.4}$$

where $\Delta_{\Lambda,i}$ is the Laplacian in the variable x_i with 0-Dirichlit boundary conditions on the boundary $\partial\Lambda$ of Λ and the interaction energy of n particles at the point $(x)_n = (x_1, x_2, ..., x_n)$ is given by

$$U((x)_n) = \sum_{1 \leqslant i < j \leqslant n} \Phi(x_i - x_j) \tag{2.1.5}$$

where Φ is a two-body potential between particles. If the interaction satisfies the stability condition

$$U((x)_n) \geqslant -Bn, \qquad B > 0$$

for all n and $(x)_n \in \Lambda^n$, $H_\Lambda^{(n)}$ is a self-adjoint operator on $\mathscr{H}_n^{(s)}(\Lambda)$, and so the total Hamiltonian H_Λ is a self-adjoint operator on $\mathsf{F}^{(s)}(\Lambda)$.[1,13]

Let $B \subset \Lambda$ be a bound region in $\mathbf{R}^\nu$. We define the local number operators N_B by

$$N_B \mid_{\mathscr{H}_n^{(s)}(\Lambda)} \psi(x_1, ..., x_n) = \sum_{i=1}^{n} \chi_B(x_i)\, \psi(x_1, ..., x_n) \tag{2.1.6}$$

where χ_B is the characteristic function of B. We note that for any $\psi \in \mathscr{H}_n^{(s)}(\Lambda)$, $N_\Lambda \psi = n\psi$. The partition function and the finite volume Gibbs states are defined by

$$\Xi_\Lambda = \mathrm{Tr}_{\mathsf{F}^{(s)}(\Lambda)}(\exp[-\beta(H_\Lambda - \mu N_\Lambda)]) \tag{2.1.7}$$

and

$$\rho_\Lambda(A) = \Xi_\Lambda^{-1}\, \mathrm{Tr}_{\mathsf{F}^{(s)}(\Lambda)}(A \exp[-\beta(H_\Lambda - \mu N_\Lambda)]) \tag{2.1.8}$$

respectively, for any $\beta \in \mathbf{R}^+$, $\mu \in \mathbf{R}$, and for any bounded operator A on $\mathsf{F}^{(s)}(\Lambda)$. The above are well defined under the stability condition for any bounded $\Lambda \subset \mathbf{R}^\nu$.[1,13]

2.2. The assumptions on the Interactions and the Main Results

We now list the assumptions on the interactions. We shall assume that the interaction between particles is given by a pair potential Φ:

$$U(x_1, ..., x_n) = \sum_{1 \leqslant i < j \leqslant n} \Phi(x_i - x_j)$$

where Φ is a Lebesgue measurable function which satisfies $\Phi(x) = \Phi(-x)$ and which may take real values and the value $+\infty$. For every $r \in Z^\nu$ we define a half-open cube with side 1,

$$Q(r) = \{x \in \mathbf{R}\colon (r^i - \tfrac{1}{2}) \leqslant x^i < (r^i + \tfrac{1}{2})\} \tag{2.2.1}$$

These cubes form a partition of $\mathbf{R}^\nu$. If $X=(x_1,..., x_m)\in(\mathbf{R}^\nu)^m$, we let $n(X, r)$ be the number of points of the sequence $X=(x_1,..., x_m)$ which belong to $Q(r)$.

Assumption A. (a) *Superstability.* There exist $A>0$, $B>0$ such that if $\mathscr{R}$ is a finite subset of Z^ν and

$$x_1,..., x_m \in \bigcup_{r\in\mathscr{R}} Q(r), \qquad X=(x_1,..., x_m)$$

then

$$U(X)\geqslant \sum_{r\in\mathscr{R}} [An(X, r)^2 - Bn(X, r)] \tag{2.2.2}$$

(b) *Strong Lower Regularity.* There is a positive decreasing function ψ on $(0, +\infty)$ such that

$$\int_0^\infty t^{\nu+\mu-1}\psi(t)\, dt<\infty \tag{2.2.3}$$

where $\mu>1/2$, and for any $x\in\mathbf{R}^\nu$

$$\Phi(x)\geqslant -\psi(|x|)$$

Remark. (a) In Ref. 10 we have assumed a stronger condition than superstability, namely, the strong superstability:

$$U(X)\geqslant \sum_{r\in\mathscr{R}} [An(X, r)^{p'} - Bn(X, r)]$$

with

$$\frac{p'}{p'-1}<1+\frac{2(p'-1)}{\nu p'+(2-\nu)}$$

Thus our results in this paper are improvements of those in Ref. 10.

(b) If Φ is a positive measurable function with the property that there exist constants $c>0$ and $d>0$ such that $\Phi(x)>c$ for $|x|\leqslant d$, then Assumption A is automatically satisfied.[13,14]

We first give our main estimate:

Proposition 2.2.1. Let the interaction satisfy Assumption A and let $\nu<4$ if the potential is not repulsive. For given $\beta\in(0,\infty)$ and $\mu\in\mathbf{R}$, let ρ_Λ be the finite volume Gibbs states defined in (2.1.7) for the interaction. Then for any $B\subset\Lambda$ and $a\in\mathbf{R}$ there is a constant $A(B, a)$ such that

$$\rho_\Lambda(\exp\{aN_B\})\leqslant \exp A(B, a)$$

holds, where $A(B, a)$ depends only on $\operatorname{diam}(B)$ and $|a|$.

We will produce the proof of the above proposition in Sections 3 and 4.

We next consider some consequences of Proposition 2.2.1. We introduce the CCR algebra of the local observables. For the details we refer to Refs. 1 and 13. For $f \in L^2(\Lambda, d^\nu x)$, let $a(f)$ and $a(f)^*$ be the annihilation and creation operators defined on $\mathsf{F}^{(s)}(\Lambda)$. Then $\Phi(f) = (1/\sqrt{2})[a(f) + a(f)^*]$ for real f has a self-adjoint extension, which we write again $\Phi(f)$. We write the Weyl operator by $W(f) = \exp[i\Phi(f)]$, and let $\mathsf{A}(\Lambda)$ be the C^* algebra generated by the Weyl operator $W(f), f \in L^2(\Lambda, d^\nu x)$, and let

$$\mathsf{A} = \overline{\bigcup_{\Lambda \subset \mathbf{R}^\nu} \mathsf{A}(\Lambda)} \tag{2.2.4}$$

be the quasi local CCR algebra in sense of Ref. 1. To study the infinite volume equilibrium states we follow the Green's function method used in Chapter 6.3 of Ref. 1. Let α^Λ be the time evolution automorphism on A_Λ given by

$$\alpha_t^\Lambda(B) = e^{itH_\Lambda} B e^{-itH_\Lambda} \tag{2.2.5}$$

We then define the finite volume Green's functions by

$$G_\Lambda(A, B; t) = \rho_\Lambda(A \alpha_t^\Lambda(B)) \tag{2.2.6}$$

Although ρ_Λ is defined as a state over A_Λ, it has an extension to a state on A by the Hahn–Banach theorem which we denote again ρ_Λ. The bounds

$$|G_\Lambda(A, B; t)| \leqslant \|A\| \, \|B\| \tag{2.2.7}$$

imply that there exists a subnet $\{\Lambda_\alpha\}$ such that

$$G(A, B; t) = \lim_{\Lambda_\alpha \to \mathbf{R}^\nu} G_\Lambda(A, B; t) \tag{2.2.8}$$

for all $A, B \in \mathsf{A}$, $t \in \mathbf{R}$. This is a consequence of Tychonoff's theorem. Clearly the value

$$\rho(A) = G(A, 1; 0) \tag{2.2.9}$$

determines a state ρ over the quasilocal algebra A.

Theorem 2.2.2. Under the assumptions stated in Proposition 2.2.1, any weak* limit ρ of ρ_Λ defines an entire analytic state over the CCR algebra A. The state ρ has finite local particle density and hence is locally normal.

Proof. Using Proposition 2.2.1 and following the method used in the proof of Theorem II.3.2 of Ref. 10 one may get the bounds of the form

$$\rho_A\left(\prod_{i=1}^{k} N_{B_i}\right) \leqslant ck! \prod_{i=1}^{k} N^+(B_i) \tag{2.2.10}$$

where $N^+(B)$ is the minimal numner of unit cubes which cover B. The theorem follows from (2.2.10) and from the method used in the proof of Theorem II.3.3 of Ref. 10. See also Section 6.3 of Ref. 1.

We next construct an infinite volume physical Hilbert space, a strongly continuous time evolution group of unitary operators, and a time translation invariant vector. Let the two-body potential Φ satisfy the following estimate: For some $\varepsilon > 0$

$$\int d^\nu x (1 + |x|^2)^{\nu+\varepsilon} |\Phi(x)|^2 < \infty \tag{2.2.11}$$

Let $G(A, B; t)$ be the Green's functions defined in (2.2.8). From the bound (2.2.10), (2.2.11) and the methods used in the proofs of Proposition 6.3.29 and Theorem 6.3.31 of Ref. 1, it follows that G: $\mathsf{A} \times \mathsf{A} \times \mathbf{R} \to \mathbf{C}$ satisfies the following properties:

(1) $A, B \to G(A, B; t)$ is bilinear for all $t \in \mathbf{R}$.
(2) $t \to G(A, B; t)$ is continuous for all $A, B \in \mathsf{A}$.
(3) $G(A, CB; 0) = G(AC, B; 0)$ for all $A, B, C \in \mathsf{A}$.
(4) $G(\mathbf{1}, \mathbf{1}; 0) = 1$.
(5) $\sum_{i,j=1}^{n} G(A_i^*, A_j; t_j - t_i) \geqslant 0$ for any finite sequences $\{A_i\}_{i=1}^n$ in A and $\{t_i\}_{i=1}^n$ in $\mathbf{R}$.
(6) [Weak KMS conditions] For all $A, B \in \mathsf{A}$, and for all $\hat{f} \in \mathsf{D}$ $\int dt\, f(t)\, G(A, B; t) = \int dt\, f(t + i\beta)\, G(B, A; -t)$.

The following result follows from Theorems 6.3.27 and 6.3.28 of Ref. 1.

Theorem 2.2.3. Let the interaction satisfy Assumption A and let $\nu < 4$ if the potential is not repulsive. Assume that the potential satisfies the bound in (2.2.11). Let $(\mathscr{H}_\rho, \pi_\rho, \Omega_\rho)$ be the cyclic representation of A with respect to the state defined in (2.2.9). Then there exists a Hilbert space $\mathscr{H}$ containing $\mathscr{H}_\rho$, a strongly continuous representation U of $\mathbf{R}$ such that

(i) $\mathscr{H} = \bigvee_{t \in \mathbf{R}} U(t)\, \mathscr{H}_\rho$;
(ii) $G(A, B; t) = (\pi_\rho(A^*)\, \Omega_\rho, U(t)\, \pi_\rho(B)\, \Omega_\rho)$, $A, B \in \mathsf{A}$, $t \in \mathbf{R}$;
(iii) ρ is a modular state on A, i.e., Ω_ρ is separating $\pi_\rho(\mathsf{A})''$.

Furthermore (i) and (ii) determine $(\mathscr{H}, U)$ uniquely up to unitary equivalence.

Since the proof of the theorem is same as those in the proofs of Theorems 6.3.27, 6.3.28, and 6.3.31 of Ref. 1, we do not produce the proof here and refer to Ref. 1.

Finally we discuss the existence of the infinite volume limit pressure and its independence of boundary conditions. Let $\Delta_{\sigma,\Lambda}$ be the Laplacian with boundary conditions $\partial\psi/\partial n = \sigma\psi$ where $\sigma \in C^1(\partial\Lambda)$. Dirichlet boundary conditions are corresponded to $\sigma = \infty$. In this case we have written $\Delta_{\infty,\Lambda}$ by Δ_Λ. Neumann boundary conditions are corresponded to $\sigma = 0$. Let $\Xi_{\sigma,\Lambda}$ be partition functions obtained by replacing Δ_Λ by $\Delta_{\sigma,\Lambda}$ in the definition Ξ_Λ in (2.1.7). Then, it follows from the minimax theorem that

$$\Xi_\Lambda = \Xi_{\infty,\Lambda} \leqslant \Xi_{\sigma,\Lambda} \leqslant \Xi_{0,\Lambda} \tag{2.2.12}$$

For the detailed discussion on boundary conditions, we refer to Section 6.3 of Ref. 1.

The finite volume pressure is defined by

$$P_{\sigma,\Lambda} = \frac{1}{|\Lambda|} \log \Xi_{\sigma,\Lambda} \tag{2.2.13}$$

We now state the results on the pressure:

Theorem 2.2.4. Let the interaction satisfy Assumption A. Assume that the two-body potential Φ satisfies the following decay property: there is a positive function $\tilde{\varphi}$ on $(0, \infty)$ such that $|\Phi(|x|)| \leqslant \tilde{\varphi}(|x|)$ for $|x| \geqslant b$ for some $b > 0$ and

$$\int_b^\infty \tilde{\varphi}(t)\, t^{\nu-1}\, dt < \infty \tag{2.2.14}$$

Then the infinite volume limit pressure

$$P_\sigma = \lim_{\Lambda \uparrow \mathbf{R}^\nu} P_{\sigma,\Lambda}$$

exists for each $0 \leqslant \sigma \leqslant \infty$ as Λ tends to $\mathbf{R}^\nu$ in the sense of van Hove. Furthermore the limit P_σ is independent of boundary conditions σ.

We postpone the proof of the above theorem to Section 5. We believe that one may be able to remove the condition on the decay property of potentials in (2.2.14) by improving the method used in Section 5 (the proof of Proposition 5.1).

3. BOUNDS ON LOCAL NUMBER OPERATORS

3.1. The Wiener Integral Formalism and the Decompositions

In this section and Section 3.1 we prove Proposition 2.2.1 under an assumption of one estimate (Theorem 3.1.1). The methods we will use are the Wiener integral formalism in statistical mechanics and the method similar to that used in Section IV of Ref. 10. We will modify the method used in Ref. 10 to combine the main ideas of Ref. 2 so that the stronger conditions in Ref. 10 are removed.

We start by introducing the Wiener integral formalism. For the details we refer to Refs. 1, 6, and 7. We will use the following notation:

$$(x)_n = (x_1,..., x_n), \qquad x_i \in \mathbf{R}^\nu$$
$$d(x)_n = \prod_{i=1}^{n} d^\nu x_i \tag{3.1.1}$$

The path space of the Wiener measure can be choosen to be

$$\Omega = C([0, \beta], \mathbf{R}^\nu)$$

The Wiener measure $P^\beta(x, y; d\omega)$, conditioned on those paths $\omega \in \Omega$ with $\omega(0) = x$, $\omega(\tau = \beta) = y$, is a σ-additive, finite measure on Ω. Let χ_B^β be the characteristic function of the subset $\{\omega \in \Omega: \omega(\tau) \in B \text{ for all } \tau \in [0, \beta]\}$. We will drop the superscript β from $P^\beta(x, y; d\omega)$ and χ_B^β if there is no confusion involved. We set

$$P_\Lambda(x, y; d\omega) \equiv \chi_\Lambda(\omega)\, P(x, y; d\omega)$$
$$P_\Lambda((x)_n, (y)_n; d(\omega)_n) \equiv \prod_{j=1}^{n} P_\Lambda(x_j, y_j; d\omega_j) \tag{3.1.2}$$

Let $\psi_\Lambda^\beta((x)_n, (y)_n)$ be the kernel of the operator $\exp(-\beta H_\Lambda^{(n)})$ on $L^2(\Lambda^n)$, where $H_\Lambda^{(n)}$ is the n-particle Hamiltonian given in (2.1.3). By the Feynman–Kac formular[1,6,7]

$$\psi_\Lambda^\beta((x)_n, (y)_n) = \int_{\Omega^n} P_\Lambda((x)_n, (y)_n; d(\omega)_n) \exp\left[-\int_0^\beta U((\omega(\tau))_n)\, d\tau \right] \tag{3.1.3}$$

Let S_n be the group of the permutations of $\{1, 2,..., n\}$ and let A be the multiplication operator by a function $A((x)_n)$ invariant under any $\pi \in S_n$.

Then we have

$$\operatorname{Tr}_{\mathscr{H}_n^{(s)}(\Lambda)}(A\exp[-\beta H_\Lambda^{(n)}])=\frac{1}{n!}\sum_{\pi\in S_n}\int_{\Lambda^n} d(x)_n\, A((x)_n)$$
$$\times\int_{\Omega^n} P_\Lambda((x)_n, \pi(x)_n; d(\omega)_n)\exp\left[-\int_0^\beta U((\omega(\tau)_n)\,d\tau\right] \tag{3.1.4}$$

where $\pi(x)_n=(x_{\pi(1)},..., x_{\pi(n)})$.

We next express the expectation of local number operators in terms of the Wiener integrals. From (2.1.1)–(2.1.4), (2.1.7), and (2.1.8) it follows that

$$\Xi_\Lambda=\sum_{n=0}^{\infty} z^n \operatorname{Tr}_{\mathscr{H}_n^{(s)}(\Lambda)}(\exp[-\beta H_\Lambda^{(n)}])$$
$$\rho_\Lambda(\exp(aN_B))=\Xi_\Lambda^{-1}\sum_{n=0}^{\infty} z^n \operatorname{Tr}_{\mathscr{H}_n^{(s)}(\Lambda)}(\exp(aN_B)\exp[-\beta H_\Lambda^{(n)}]) \tag{3.1.5}$$

where $z=e^{\beta\mu}$ and the term corresponding to $n=0$ equals to 1. In the rest of this paper we use the following notations;

$$U((\omega)_n)=\int_0^\beta U((\omega(\tau))_n)\,d\tau$$
$$N_B((x)_n)=\sum_{i=1}^{n}\chi_B(x_i) \tag{3.1.6}$$

From (3.1.4) it then follows that

$$\operatorname{Tr}_{\mathscr{H}_n^{(s)}(\Lambda)}(\exp[aN_B]\exp[-\beta H_\Lambda^{(n)}])=\frac{1}{n!}\sum_{\pi\in S_n}\int_{\Lambda^n} d(x)_n\exp[aN_B((x)_n)]$$
$$\times\int_{\Omega^n} P_\Lambda((x)_n, \pi(x)_n; d(\omega)_n)\, e^{-U((\omega)_n)} \tag{3.1.7}$$

By a translation one may assume that B is contained in the ball of diam(B) centered at the origin.

We now introduce a decomposition of the space of n paths into disjoint subsets. To do this we will use the following notations: For any given path configuration $(\omega)_n$ and for any given unit cube $Q(r)$ centered at $r\in Z^\nu$, let

$$n(r,\tau)=\operatorname{card}\{\omega_i\colon \omega_i\in(\omega)_n,\ \omega_i(\tau)\in Q(r)\} \tag{3.1.8}$$

That is, $n(r, \tau)$ is the number of ω_i, $i = 1, 2, ..., n$, such that $\omega_i(\tau) \in Q(r)$. For given small $\alpha > 0$ and for $q \in \mathbf{N}$, let

$$\begin{aligned} l_q &= [e^{\alpha q}] \\ \Lambda_q &= [-l_q - \tfrac{1}{2}, l_q + \tfrac{1}{2}]^\nu \\ |\Lambda_q| &= (2l_q + 1)^\nu \end{aligned} \tag{3.1.9}$$

where $[a]$ is the integer part of $a \in \mathbf{R}$. For a given $\nu \in \mathbf{N}$, let us denote

$$\gamma(\nu, 2) = 1 + \frac{2}{\nu + 2} \tag{3.1.10}$$

We now introduce a decomposition of the space of n-Wiener paths into a union of multually disjoint subsets

$$\Omega^n = \mathscr{E}_0 \cup \left(\bigcup_{q \geq q_0} \mathscr{E}_q \right) \tag{3.1.11}$$

as follows: For fixed $1 < l < p < \gamma(\nu, 2)$ which we will choose later and for a fixed large $q_0 \in Z$, let

$$F_q(p) = \sup_{\tau \in [0, \beta]} \sum_{Q(r) \subset \Lambda_q} n(r, \tau)^p \tag{3.1.12}$$

$$E_q = \sum_{Q(r) \subset \Lambda_q} \int_0^\beta n(r, \tau)^2 \, d\tau \tag{3.1.13}$$

$$\begin{aligned} \mathscr{E}_0 &= \{(\omega)_n \in \Omega^n \colon F_q(p) + E_q < |\Lambda_q|^l \text{ for all } q \geq q_0\} \\ \mathscr{E}_q &= \{(\omega)_n \in \Omega^n \colon F_q(p) + E_q \geq |\Lambda_q|^l \text{ and } F_{q'}(p) + E_{q'} < |\Lambda_{q'}|^l \\ &\qquad \text{for all } q_0 \leq q < q'\} \end{aligned} \tag{3.1.14}$$

The above decomposition is a modification of those used in Refs. 10 and 2 in the following way. In Ref. 10 we have only used the quantity $F_q(p)$ to define $\mathscr{E}_0$ and $\mathscr{E}_q$. On the other hand the authors of Ref. 2 only used the quantity E_q for the decomposition. In a sense $F_q(p)$ and E_q are L^∞ norm and L^1 norm, respectively. As mentioned in the Introduction the method used in Ref. 10 is good enough to control the quantum statistics, but not powerful enough to handle all superstable interactions. Even if the method of Ref. 2 is appropriated to deal with interaction terms, it cannot control the quantum statistics. Thus we conbine two methods as above.

We observe that for $q < q'$ and for $\tau \in [0, \beta]$

$$\begin{aligned} \sum_{Q(r) \subset \Lambda_{q'}} n(r, \tau) &\leq |\Lambda_{q'}| \left(|\Lambda_{q'}|^{-1} \sum_{Q(r) \subset \Lambda_{q'}} n(r, \tau)^p \right)^{1/p} \\ &\leq |\Lambda_{q'}|^{1 + (l-1)/p} \qquad \text{on} \quad \mathscr{E}_q \end{aligned} \tag{3.1.15}$$

by Holder's inequality and by the definition of $\mathscr{E}_q$ in (3.1.13) and (3.1.14). Also we have that for $q < q'$

$$\sum_{Q(r) \subset \Lambda_q} \int_0^\beta n(r, \tau)^2 \, d\tau < |\Lambda_{q'}|^l \qquad \text{on} \quad \mathscr{E}_q \tag{3.1.16}$$

The bounds in (3.1.15) and (3.1.16) will be used later repeatly.

Let $\mathscr{E}_{q,k}$ be the subset of $\mathscr{E}_q$ in which k paths hit Λ_{q+1} during the time interval $[0, \beta]$:

$$\mathscr{E}_{q,k} = \{(\omega)_n \in \mathscr{E}_q : \text{card}(\{\omega_i : \omega_i(\tau) \in \Lambda_{q+1} \text{ for some } \tau \in [0, \beta]\}) = k\} \tag{3.1.17}$$

and let $\bar{\mathscr{E}}_{q,k}$ be the subset of $\mathscr{E}_q$ in which $\omega_1, \ldots, \omega_k$ hit Λ_{q+1} during $[0, \beta]$:

$$\bar{\mathscr{E}}_{q,k} = \{(\omega)_n \in \mathscr{E}_{q,k} : \omega_i(\tau) \in \Lambda_{q+1} \text{ for some } \tau \in [0, \beta], \, i = 1, \ldots, k\} \tag{3.1.18}$$

Then we have

$$\mathscr{E}_q = \bigcup_k \mathscr{E}_{q,k} \tag{3.1.19}$$

From the decompositions in (3.1.12) and (3.1.19) and from (3.1.5) and (3.1.7) it follows that

$$\rho_\Lambda(\exp[aN_B]) = G_0 + \sum_{n=0}^{\infty} \sum_{q \geqslant q_0} \sum_k G^{(n)}(q, k) \tag{3.1.20}$$

where G_0 is the contribution from $\mathscr{E}_0$:

$$G_0 = \Xi_\Lambda^{-1} \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\pi \in S_n} \int_{\Lambda^n} d(x)_n \exp[aN_B((x)_n)]$$
$$\times \int_{\mathscr{E}_0} P_\Lambda((x)_n, \pi(x)_n; d(\omega)_n) \, e^{-U((\omega)_n)} \tag{3.1.21}$$

and $G^{(n)}(q, k)$ is the contribution from $\mathscr{E}_{q,k}$;

$$G^{(n)}(q, k) = \Xi_\Lambda^{-1} \frac{z^n}{n!} \sum_{\pi \in S_n} \int_{\Lambda^n} d(x)_n \exp[aN_B((x)_n)]$$
$$\times \int_{\mathscr{E}_{q,k}} P_\Lambda((x)_n, \pi(x)_n; d(\omega)_n) \exp[-U((\omega)_n)] \tag{3.1.22}$$

For a given $B \subset \Lambda$ we choose q_0 sufficiently large so that $B \subset \Lambda_{q_0}$. Then by the bound in (3.1.15) we have that for $q_0 \leqslant q$

$$N_B((x)_n) \leqslant \sum_{Q(r) \in \Lambda_{q+1}} n(r, \tau = 0) \leqslant |\Lambda_{q+1}|^{1 + (l-1)/p} \qquad \text{on} \quad \mathscr{E}_q \tag{3.1.23}$$

and so

$$G_0 \leqslant \exp[|a|\ |\Lambda_{q_0}|^{1+(l-1)/p}]$$
$$G^{(n)}(q,k) \leqslant \Xi_\Lambda^{-1} z^n \exp[|a|\ |\Lambda_{q+1}|^{1+(l-1)/p}]\ \bar{G}^{(n)}(q,k) \tag{3.1.24}$$

where

$$\bar{G}^{(n)}(q,k) = \frac{1}{n!}\int_{\Lambda^n} d(x)_n \int_{\mathscr{E}_{q,k}} P_\Lambda((x)_n, \pi(x)_n; d(\omega)_n)\, e^{-U((\omega)_n)} \tag{3.1.25}$$

We reindex the k paths ω_i's which hit Λ_{q+1} so that ω_i, $i=1, 2.,,,. k$ hit Λ_{q+1}. There are $n!/(n-k)!\,k!$ ways to choose k paths among n paths and so

$$\bar{G}^{(n)}(q,k) = \frac{1}{(n-k)!\,k!} \sum_{\pi\in S_n} \int_{\Lambda^n} d(x)_n \times \int_{\bar{\mathscr{E}}_{q,k}} P_\Lambda((x)_n, \pi(x)_n; d(\omega)_n)\, e^{-U((\omega)_n)} \tag{3.1.26}$$

Notice that we have $(n-k)!$ factor instead of $n!$. The total number of terms in the above is card $(S_n) = n!$. This is the main problem arising from quantum statistics.

For any configulation $\{\omega_i\}$ of paths in $\bar{\mathscr{E}}_{q,k}$ we write

$$U_1((\omega)_k) = \int_0^\beta U_1((\omega(\tau))_k)\, d\tau$$
$$U_2((\omega)_{n-k}) = \int_0^\beta U_2((\omega(\tau))_{n-k})\, d\tau \tag{3.1.27}$$
$$W((\omega)_k, (\omega)_{n-k}) = \int_0^\beta W((\omega(\tau))_k, (\omega(\tau))_{n-k})\, d\tau$$

where

$$U_1((\omega(\tau))_k) = \sum_{1\leqslant i<j\leqslant k} \Phi(\omega_i(\tau) - \omega_j(\tau))$$
$$U_2((\omega(\tau))_{n-k}) = \sum_{k+1\leqslant i<j\leqslant n} \Phi(\omega_i(\tau) - \omega_j(\tau)) \tag{3.1.28}$$
$$W((\omega(\tau))_k, (\omega(\tau))_{n-k}) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \Phi(\omega_i(\tau) - \omega_j(\tau))$$

We then have

$$U((\omega)_n) = U_1((\omega)_k) + U_2((\omega)_{n-k}) + W((\omega)_k, (\omega)_{n-k}) \quad (3.1.29)$$

As in Ref. 10 we define the fluctuations of k paths as follows: we write

$$V(\omega) = \sup_{\tau_1, \tau_2 \in [0,\beta]} |\omega(\tau_1) - \omega(\tau_2)|^2$$
$$V((\omega)_k) = \sum_{i=1}^{k} V(\omega_i) \quad (3.1.30)$$

The following is the main technical estimate in proving Prososition 2.2.1:

Theorem 3.1.1. Let the interaction satisfy Assumption A and let $v < 4$ if the potential is not repulsive. For a given $b > 0$ one can choose α in (4.1.9) small enough, q_0 large enough, and $1 < l < p < \gamma(v, 2)$ such that there exist constant $c > 0$ and $\varepsilon > 0$ such that

$$U_1((\omega)_k) + bV((\omega)_k) + W((\omega)_k, (\omega)_{n-k}) \geqslant c(|A_q|^{1+[(l-1)(1+\varepsilon)/p]} + k^{1+\varepsilon})$$

on $\bar{\mathscr{E}}_{q,k}$.

The above theorem is the result corresponding to Theorem IV.2.1 of Ref. 10. We will prove the theorem in Section 4.

3.2. The Proof of Proposition 2.2.1; Control of Quantum Statistics

In this section we prove Proposition 2.2.1 under the assumption that Theorem 3.1.1 holds. The proof of Proposition 2.2.1 is almost identical to that in Section IV.3 of Ref. 10. In Ref. 10 the expression in Theorem 3.1.1 is bounded by $|A_{q+1}|^{1+(l-1)\gamma/p}$ for some $\gamma > 1$ (see Theorem IV.3.1 of Ref. 10) and also k is bounded by $|A_{q+1}|^{1+(l-1)/p}$ on $\bar{\mathscr{E}}_{q,k}$. Here we do not have such a bound for k. But we have the convergent factor $k^{1+\varepsilon}$ in Theorem 3.1.1. Thus, if one redefine M_γ in (4.3.26) of Ref. 10 by $M_\gamma = k + 2\,|A_{q+\alpha^{-1}\log(\gamma+3)}|^{1+(l-1)/P}$, every argument used in Section IV.3 can be applied to our cases here. Therefore in principle the proof of Proposition 2.2.1 is complete. Since we try to make this paper as self-contained as possible, we will prove Proposition 2.2.1 in more details by following every necessary steps used in Section IV.3 of Ref. 10.

From Theorem 3.1.1 and from (3.1.26) and (3.1.29) it follows that

$$\bar{G}^{(n)}(q, k) \leqslant \exp[-c(|A_q|^{1+(l-1)(1+\varepsilon)/p} + k^{1+\varepsilon})]\, \bar{F}^{(n)}(q, k) \quad (3.2.1)$$

where

$$\bar{F}^{(n)}(q,k)=\frac{1}{(n-k)!\,k!}\sum_{\pi\in S_n}\int_{\Lambda^n}d(x)_n\int_{\bar{\mathscr{E}}_{q,k}}P_\Lambda((x)_n,\pi(x)_n;d(\omega)_n)$$
$$\times\exp[bV((\omega)_k)-U_2((\omega)_{n-k})] \tag{3.2.2}$$

As we have discussed before, there are $n!$ terms in (3.2.2). Therefore one has to show that many terms in (3.2.2) do not contribute. The main idea is decoupling of the paths in $\{\omega_i: i=1,..., k\}$ from the paths in $\{\omega_i: i=k+1,..., n\}$. On the subset $\bar{\mathscr{E}}_{q,k}$ a path ω_i, $i\leqslant k$, may join to a path ω_j, $k<j$, to form a composite path, i.e., $x_{\pi(i)}=x_j$. If the fluctuation of ω_i is small, there will not be many ways to form a composite tragectory. On the other hand, if the fluctuation is large, the corresponding Wiener measure will be small. Thus we further decompose the subset $\bar{\mathscr{E}}_{q,k}$ into a union of mutually disjoint subsets corresponding to large and small fluctuations.

We adapt the notations used in Section III.2 and Section IV.3 of Ref. 10. For a unit cube Δ and for $l\geqslant 0$, we write

$$\Lambda(\Delta,l)=\{x\in\mathbf{R}^\nu: \max_{1\leqslant i\leqslant\nu}\inf_{y\in\Delta}|x^i-y^i|\leqslant l\}$$

That is, $\Lambda(\Delta, l)$ is the cube with its volume $(2l+1)^\nu$ containing Δ at the center. We denote

$$\mathscr{E}_{\Delta,l}=\{\omega\in\Omega;\omega(\tau)\in\Delta,\ \omega(\tau')\in\Lambda(\Delta,l+1)\backslash\Lambda(\Delta,l)\text{ for some }\tau,\tau'\in[0,\beta],$$
$$\text{and }\omega(\tau'')\notin\Lambda(\Delta,l+1)^c\text{ for all }\tau''\in[0,\beta]\} \tag{3.2.3}$$
$$\mathscr{E}_{\Delta,-1}=\{\omega\in\Omega:\omega(\tau)\in\Delta\text{ for all }\tau\in[0,\beta]\}$$

where A^c is the complement of A. Note that any path in $\mathscr{E}_{\Delta,l}$, $l\geqslant 0$, must visit Δ and $\Lambda(\Delta, l+1)\backslash\Lambda(\Delta, l)$, but not $\Lambda(\Delta, l+1)^c$. We also have that for any positive function F on Ω

$$\int P_\Lambda(x,y;d\omega)[1-\chi_{\Lambda^c_{q+1}}(\omega)]\,F(\omega)$$
$$\leqslant\sum_{\Delta\subset\Lambda_{q+1}}\sum_{l=-1}^{\infty}\int_{\mathscr{E}_{\Delta,l}}P(x,y;d\omega)\,F(\omega) \tag{3.2.4}$$

where $\sum_{\Delta\subset\Lambda_{q+1}}$ is the summation over unit cube $\Delta=Q(r)\subset\Lambda_{q+1}$. That is, in order to hit Λ_{q+1} the path must hit one of $Q(r)\subset\Lambda_{q+1}$. For the detailed derivation of (3.2.4) we refer to the derivation of (4.3.7) in Ref. 10. Let $\mathscr{E}^{(i)}_{\Delta,l}$ be the subset $\mathscr{E}_{\Delta,l}$ for the i path, $i=1,..., k$. We write

$$\bar{\mathscr{E}}(q,k;(\Delta_1,l_1),...,(\Delta_k,l_k))$$
$$=\{(\omega)_n\in\bar{\mathscr{E}}_{q,k};\omega_i\in\mathscr{E}^{(i)}_{\Delta_i,l_i},\ i=1,...,k\} \tag{3.2.5}$$

Since the fluctuation is bounded by $V(\omega) \leqslant [\operatorname{diag}(\Lambda(\Delta, l+1))]^2 \leqslant 4v(l+2)^2$ on $\mathscr{E}_{\Delta,l}$, we have that on $\bar{\mathscr{E}}(q, k; (\Delta_1, l_1),..., (\Delta_k, l_k))$

$$V((\omega)_k) \leqslant \sum_{i=1}^{k} (l_i+2)^2\, 4v \tag{3.2.6}$$

We remember that ω_i, $i \leqslant k$, must visit Λ_{q+1} at least one time. Thus we have

$$\begin{aligned} &\int_{\bar{\mathscr{E}}_{q,k}} P_\Lambda((x)_n, (y)_n; d(\omega)_n) \cdots \\ &\quad = \int_{\bar{\mathscr{E}}_{q,k}} P_\Lambda((x)_n, (y)_n; d(\omega)_n) \prod_{i=1}^{k} [1 - \chi^c_{\Lambda_{q+1}}(\omega_i)] \cdots \end{aligned} \tag{3.2.7}$$

We apply (3.2.7), (3.2.4), and (3.2.6) (in that order) to (3.2.2) to obtain the bound

$$\begin{aligned} \bar{F}^{(n)}(q, k) \leqslant &\frac{1}{(n-k)!\, k!} \sum_{\Delta_1 \subset \Lambda_{q+1}} \cdots \sum_{\Delta_k \subset \Lambda_{q+1}} \sum_{l_1=-1}^{\infty} \cdots \sum_{l_k=-1}^{\infty} \\ &\times \exp\Big[4vb \sum_{i=1}^{b} (l_i+2)^2\Big]\, \bar{F}^{(n)}(q, k; (\Delta_1, l_1),..., (\Delta_k, l_k)) \end{aligned} \tag{3.2.8}$$

where

$$\begin{aligned} \bar{F}^{(n)}(q, k; (\Delta_1, l_1),..., (\Delta_k, l_k)) &= \sum_{\pi \in S_n} \int_{\Lambda^n} d(x)_n \mathbf{K}((x)_n, \pi(x)_n) \\ \mathbf{K}((x)_n, (y)_n) &\equiv \int_{\bar{\varepsilon}(q,k;(\Delta_1,l_1),...,(\Delta_k,l_k))} \\ &\quad \times P_\Lambda((x)_n, (y)_n; d(\omega)_n)\, e^{-U_2((\omega)_{n-k})} \end{aligned} \tag{3.2.9}$$

We recall that for any configuration of n path $(\omega)_n$ in $\bar{\mathscr{E}}(q, k; (\Delta_1, l_1),..., (\Delta_k, l_k))$ the path ω_i, $i = 1,..., k$, must stay inside of $\Lambda(\Delta_i, l_i+1)$ by (3.2.3) and (3.2.5). This means that ω_i, $i=1,..., k$, cannot form a composite path with any path ω_l if $\omega_l(\tau=0) \in \Lambda(\Delta_i, l_i+1)^c$. By the bound (3.1.15) there are at most $|\Lambda_{q+\alpha^{-1}\log(l_i+3)}|^{1+(l-1)/p}$ number of paths which can hit $\Lambda(\Delta, l_i+1) \subset \Lambda_{q+\alpha^{-1}\log(l_i+3)}$. Thus if l_i, $i \leqslant k$, are small, many terms in (3.2.9) will be vanished. If l_i's are large, the corresponding Wiener integral will be small (see Proposition 3.2.3 in the following). This is the idea of controlling quantum statistics.

We will use the following abbreviated notations:

$$M(l_i) = |\Lambda_{q+\alpha^{-1}\log(l_i+3)}|^{1+[l-1)/p]}$$
$$M_{l_i} = [2M(l_i)] + k \tag{3.2.10}$$
$$\gamma = \max\{l_1,..., l_k\}$$

We remark that the definition of M_γ differ to that in (4.3.26) of Ref. 10 by the factor k. We will use the fact that $M_\gamma - k > M(\gamma)$ as in Ref. 10. We write

$$S(k, n) = \{\pi \in S_n : \pi \text{ is a permutation of } \{k+1,..., n\}\}$$
$$E(m, m') = \{\pi \in S_n : \pi \text{ is an interchange of } m \text{ with one of}$$
$$m, m+1,..., m' \text{ for } m \leqslant m' \leqslant n\}$$
$$E(m, m') = E(m, n) \text{ if } m' > n \tag{3.2.11}$$
$$E_k(l_1,..., l_k) = \{p = p_k p_{k-1} \cdots p_1;\ p_i \in E(i, M_{l_i})\}$$
$$\tilde{E}_k(l_1,..., l_k) = \{p = p_1 p_2 \cdots p_k;\ p_i \in E(i, M_{l_i})\}$$
$$\tilde{E}_{n,k} = \{p = p_1 p_2 \cdots p_k;\ p_i \in E(i, n) \text{ for } i = 1,..., k\}$$

Here we have modified the definition of $E_k(l_1,..., l_k)$ in (4.3.14) of Ref. 10 by replacing $E(i, [2M(l_i)])$ by $E(i, M_{l_i})$. We note that

$$S_n = \tilde{E}_{n,k} S(k, n)$$

and

$$\operatorname{card}(\tilde{E}_k(l_1,..., l_k)) = \operatorname{card}(\boldsymbol{E}_k(l_1,..., l_k))$$
$$\leqslant \prod_{i=1}^{k} M_{l_i} \tag{3.2.12}$$

For any $\sigma \in S(k, n)$ we use the notation

$$\chi_\sigma : \text{the characteristic function of the subset}$$
$$\{(x)_n \in \mathbf{R}^{\nu n}: \|x_{\sigma(k+1)}\| \leqslant \|x_{\sigma(k+2)}\| \leqslant \cdots \leqslant \|x_{\sigma(n)}\|\} \tag{3.2.13}$$

where $\|x\| = = \max_{1 \leqslant i \leqslant \nu} |x^i|$. We then have

$$\sum_{\sigma \in S(k,n)} \chi_\sigma((x)_n) = \sum_{\sigma \in S(k,n)} \chi_e(\sigma(x)_n) = 1 \tag{3.2.14}$$

where $e \in S(k, n)$ is the identity element. The following is the result corresponding to Lemma 4.3.1 of Ref. 10.

Lemma 3.2.1. Let $\mathbf{K}((x)_n, (y)_n)$ be defined as in (3.2.9). Then

$$\bar{F}^{(n)}(q, k; (\Delta_1, l_1), ..., (\Delta_k, l_k))$$
$$\leqslant k! \sum_{p \in \tilde{E}_k(l_1,...,l_k)} \sum_{\sigma,\sigma' \in S(k,n)} \int_{\Lambda_n} d(x)_n \chi_e((x)_n) \, \mathbf{K}(\sigma'(x)_n, p\sigma(x)_n)$$

Proof. The lemma follows from the method used in the proof of Lemma 4.3.1 of Ref. 10, $\Lambda(\Delta, l) \subset \Lambda_{q+\alpha^{-1}\log(l+3)}$ for $\Delta \subset \Lambda_{q+1}$, and from the fact that $M_{l_i} - k > M(l_i)$. For the details, see the proof of Lemma 4.3.1 of Ref. 10.

We now briefly review the partial symmetry spaces introduced in Section 3.2 of Ref. 10. Let A_n be a subset of S_n. The subset A_n is not necessary subgroup of S_n. Let $\{f_i\}$ be an orthonormal base for $L^2(\Lambda, d^\nu x)$ and let $\mathscr{H}^{(s)}_{A_n}(\Lambda)$ be the closed subspace of $L^2(\Lambda^n)$ spanned by the following vectors:

$$\text{card}\,(A_n)^{-1} \sum_{\pi \in A_n} f_{i_1}(x_{\pi(1)}) \, f_{i_2}(x_{\pi(2)}) \cdots f_{i_n}(x_{\pi(n)})$$

Let $P(A_n)$ be the projection operator onto $\mathscr{H}^{(s)}_{A_n}(\Lambda)$. For any $f \in L^2(\Lambda^n)$ we have

$$(P(A_n) \, f)(x)_n = \text{card}(A_n)^{-1} \sum_{\pi \in A_n} f(\pi(x)_n)$$

If A be an operator of the trace class which admits an integral kernel $A((x)_n, (y)_n)$, it follows that

$$\text{Tr}_{L^2(\Lambda^n)}(P(A_n) \, AP(A_n))$$
$$\leqslant \text{card}(A_n)^{-2} \sum_{\pi,\pi' \in A_n} \int_{\Lambda^n} d(x)_n \, A(\pi^{-1}(x)_n, \pi'^{-1}(x)_n) \quad (3.2.15)$$

For more detaily we refer to Ref. 10.

Let $P(S(k, n))$ and $P(E_k(l_1, ..., l_k))$ be the projection operators onto $\mathscr{H}^{(s)}_{S(k,n)}(\Lambda)$ and $\mathscr{H}^{(s)}_{E_k(l_1,...,l_k)}(\Lambda)$, respectively. For any $\sigma \in S(k, n)$, let χ_σ be the (projection) operator defined by

$$(\chi_\sigma f)(x)_n = \chi_e(\sigma(x)_n) \, f((x)_n)$$

for any $f \in L^2(\mathbf{R}^{\nu n})$. From (3.2.13) we have

$$\sum_{\sigma \in S(k,n)} \chi_\sigma = 1 \quad (3.2.16)$$

In order to avoid notational complications we use the following abbreviated notations:

$$\begin{aligned} P_1 &= P(S(k, n)) \\ P_2 &= P(E_k(l_1,..., l_k)) \qquad (3.2.17) \\ C_{1,2} &= \mathrm{card}(S(k, n))^2 \,\mathrm{card}(E_k(l_1,..., l_k)) \end{aligned}$$

Let $\mathbf{K}$ be the operator on $L^2(\Lambda^n)$ defined by its kernel $\mathbf{K}((x)_n, (y)_n)$, where $\mathbf{K}((x)_n, (y)_n)$ has been defined in (3.2.9). Using (3.2.15) the following is easy to derive:

$$\mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 \mathbf{K} P_1 P_2 \chi_e) = C_{1,2}^{-1} \sum_{p \in \tilde{E}_k(l_1,...,l_k)} \sum_{\sigma,\sigma' \in S(k,n)} \times \int_{\Lambda^n} d(x)_n \chi_e((x)_n) \,\mathbf{K}(\sigma'(x)_n, p\sigma(x)_n) \qquad (3.2.18)$$

and so by Lemma 3.2.1 we have

$$\bar{F}^{(n)}(q, k; (\Delta_1 . l_1),..., (\Delta_k, l_k)) \leqslant k! \, C_{1,2} \,\mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 \mathbf{K} P_1 P_2 \chi_e) \qquad (3.2.19)$$

We use the identity (3.2.16) to obtain

$$\mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 \mathbf{K} P_1 P_2 \chi_e) = \sum_{\sigma'' \in S(k,n)} \mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 \mathbf{K} P_1 \chi_{\sigma''} P_2 \chi_e) \qquad (3.2.20)$$

Let γ and M_γ be defined as in (3.2.10), and let

$$A(k, n) \equiv \{\sigma'' \in S(k, n) \colon \mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 \mathbf{K} P_1 \chi_{\sigma''} P_2 \chi_e) \neq 0\} \qquad (3.2.21)$$

Then the following result corresponds to Lemma IV.3.2 of Ref. 10:

Lemma 3.2.2:

$$\mathrm{card}(A(k, n)) \leqslant \mathrm{card}(\tilde{E}(l_1,..., l_k)) \, M_\gamma!/(M_\gamma - k)!$$

Proof. The lemma follows from the method same as that used in Lemma IV.3.2 of Ref. 10 and from the fact that $M_\gamma - k > M(\gamma)$. For the details we refer to Ref. 10. ∎

Using the definition of $A(k, n)$ and (3.2.19), (3.2.20) we obtain

$$\bar{F}^{(n)}(q, k; (\Delta_1, l_1,..., (\Delta_k, l_k)) \leqslant k! \, C_{1,2} \sum_{\sigma'' \in A(k,n)} \mathrm{Tr}_{L^2(\Lambda^n)} \times (\chi_e P_1 \mathbf{K} P_1 \chi_{\sigma''} P_2 \chi_e) \qquad (3.2.22)$$

The next step is to get a bound for (3.2.22). To do this, we define

$$\mathbf{K}_{\Delta,l}(x, y) = \int_{\mathscr{E}(\Delta,l)} P(x, y; d\omega) \tag{3.2.23}$$

Let $\mathbf{K}_{\Delta,l}$ be the operator corresponding to the kernel $\mathbf{K}_{\Delta,l}(x, y)$. Let $P_{\Lambda(\Delta,l)}$ be the projection operator onto $L^2(\Lambda(\Delta, l)) \subset L^2(\mathbf{R}^\nu)$, and let

$$\bar{\mathbf{K}}_{\Delta,l} = P_{\Lambda(\Delta,l+1)} \exp\left(\frac{\beta}{4}\Delta\right) P_{\Lambda(\Delta,l+1)} \tag{3.2.24}$$

We then have the following result (Proposition III.2.2 of Ref. 10):

Lemma 3.2.3. For $l = -1, 0, 1,\ldots$ there are constants $c_1 > 0$ and $c_2 > 0$ such that

(a) $\mathbf{K}_{\Delta,l}(x, y) \leqslant c_1 \exp\left(-\frac{l^2}{16\beta}\right) \bar{\mathbf{K}}_{\Delta,l}(x, y)$

(b) $\mathrm{Tr}_{L^2(\mathbf{R}^\nu)}(\bar{\mathbf{K}}_{\Delta,l}) \leqslant c_2 |\Lambda(\Delta, l+1)|$

where $\bar{\mathbf{K}}_{\Delta,l}(x, y)$ is the kernel of $\bar{\mathbf{K}}_{\Delta,l}$ and $|\Lambda(\Delta, l+1)|$ is the volume of $\Lambda(\Delta, l+1)$.

For the proof of the above lemma we refer to Ref. 10, the above result implies that for large l (for large fluctuation) the contribution of the Wiener measure is small.

We now start to get a bound for (3.2.22). From (3.2.5) the following is obvious:

$$\bar{\mathscr{E}}(q, k; (\Delta_1, l_1),\ldots, (\Delta_k, l_k)) \subset \left(\mathop{\times}_{i=1}^{k} \mathscr{E}^{(i)}_{\Delta_i,l_i}\right) \times \Omega^{n-k} \tag{3.2.25}$$

Using (3.2.15) [and the method to get (3.2.20)] it is easy to sheck that[10]

$$\begin{aligned}\mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 \mathbf{K} P_1 \chi_{\sigma''} P_2 \chi_e) \leqslant C_{1,2}^{-1} & \sum_{p \in \tilde{E}_k(l_1,\ldots,l_k)} \sum_{\sigma',\sigma \in S(k,n)} \\ & \times \int_{\Lambda^n} d(x)_n\, \chi_e((x)_n)\, \mathbf{K}(\sigma'(x)_n, p\sigma(x)_n)\, \chi_e(p\sigma''(x)_n)\end{aligned} \tag{3.2.26}$$

From the definition of $\mathbf{K}((x)_n, (y)_n)$ in (3.2.9) and from (3.2.25) it follows that

$$\begin{aligned}\mathbf{K}((x)_n, (y)_n) \leqslant & \int_{(\times_{i=1}^{k} \mathscr{E}^{(i)}_{\Delta_i,l_i})} P((x)_k, (y)_k; d(\omega)_k) \\ & \times (\exp[-\beta H_\Lambda^{n-k}])((x)_{n-k}, (y)_{n-k})\end{aligned} \tag{3.2.27}$$

We use Lemma 3.2.3(a) to obtain

$$\mathbf{K}((x)_n, (y)_n) \leqslant \prod_{i=1}^{k} \left[c_1 \exp\left(-\frac{l_i^2}{16\beta} \right) \right] Y((x)_n, (y)_n) \tag{3.2.28}$$

$$Y((x)_n, (y)_n) \equiv \left(\prod_{i=1}^{k} \bar{\mathbf{K}}_{\Delta_i, l_i}(x_i, y_i) \right) \times (\exp[-\beta H_\Lambda^{(n-k)}])((x)_{n-k}, (y)_{n-k}) \tag{3.2.29}$$

Let Y be the operator corresponding to the kernel $Y((x)_n, (y)_n)$:

$$Y \equiv \left(\prod_{i=1}^{k} \bar{\mathbf{K}}_{\Delta_i, l_i} \right) \exp[-\beta H_\Lambda^{(n-k)}] \tag{3.2.30}$$

which is positive operator on $L^2(\mathbf{R}^{\nu n})$. From (3.2.22), (3.2.26), and (3.2.28) it follows that

$$\begin{aligned} &\bar{F}^{(n)}(q, k; (\Delta_1, l_1), \ldots, (\Delta_k, l_k)) \\ &\quad \leqslant k!\, C_{1,2} \left(\prod_{i=1}^{n} c_1 \exp\left(-\frac{l_i^2}{16\beta} \right) \right) \\ &\quad \times \sum_{\sigma'' \in A(k,n)} \operatorname{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 Y P_1 \chi_{\sigma''} P_2) \end{aligned} \tag{3.2.31}$$

We now use the positivity of Y and the abstract Hölder's inequality[12] to conclude that

$$\begin{aligned} &\operatorname{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 Y P_1 \chi_{\sigma''} P_2) \\ &\quad \leqslant [\operatorname{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 Y P_1)]^{1/2} [\operatorname{Tr}_{L^2(\Lambda^n)}(\chi_{\sigma''} P_1 Y P_1)]^{1/2} \end{aligned} \tag{3.2.32}$$

Here we have used the fact that $\operatorname{Tr}(PAP) \leqslant \operatorname{Tr}(A)$ for any projection P and positive A. A direct computation gives us that for any $\sigma'' \in (S(k, n))$:

$$\operatorname{Tr}_{L^2(\Lambda^n)}(\chi_{\sigma''} P_1 Y P_1) = [(n-k)!]^{-1} \operatorname{Tr}_{L^2(\Lambda^n)}(P_1 Y P_1) \tag{3.2.33}$$

See the below of (4.3.36) of Ref. 10 for the detailed derivation of the above equality. We substitute (3.2.33) into (3.2.32) and then we use the definitions of P_1 and Y in (3.2.17) and (3.2.30), respectively, to obtain

$$\begin{aligned}
&\mathrm{Tr}_{L^2(\Lambda^n)}(\chi_e P_1 Y P_1 \chi_{\sigma''} P_2) \\
&\quad \leqslant \frac{1}{(n-k)!} \mathrm{Tr}_{L^2(\Lambda^n)}(P_1 Y P_1) \\
&\quad = \frac{1}{(n-k)!} \left(\prod_{i=1}^{k} \mathrm{Tr}_{L^2(\Lambda)}(\bar{\mathbf{K}}_{\Delta_i, l_i}) \right) \mathrm{Tr}_{\mathscr{H}^{(s)}_{n-k}(\Lambda)}(\exp[-\beta H_\Lambda^{(n-k)}]) \\
&\quad \leqslant \frac{1}{(n-k)!} c_2^k \prod_{i=1}^{k} (2l_i + 2)^\nu \, \mathrm{Tr}_{\mathscr{H}^{(s)}_{n-k}(\Lambda)} (\exp[-\beta H_\Lambda^{(n-k)}]) \qquad (3.2.34)
\end{aligned}$$

for any $\sigma'' \in S(k, n)$. Here we have used Lemma 3.2.3(b) to get the third inequality. We now combine (3.2.31) and (3.2.34) to obtain

$$\begin{aligned}
&\bar{F}^{(n)}(q, k; (\Delta_1, l_1), \ldots, (\Delta_k, l_k)) \\
&\quad \leqslant M(q, k; l_1, \ldots, l_k) \, \mathrm{Tr}_{\mathscr{H}^{(s)}_{n-k}(\Lambda)}(\exp[-\beta H_\Lambda^{(n-k)}]) \qquad (3.2.35)
\end{aligned}$$

where

$$\begin{aligned}
&M(q, k; l_1, \ldots, l_k) \\
&\quad = \mathrm{card}(A(k, n)) \, k! \, C_{1,2} \frac{1}{(n-k)!} c_2^k \prod_{i=1}^{k} (2l_i + 2)^\nu \\
&\quad \times \exp\left(-\frac{l_i^2}{16\beta} \right) \qquad (3.2.36)
\end{aligned}$$

From the bound in Lemma 3.2.2

$$\mathrm{card}(A(k, n)) \leqslant \mathrm{card}(\tilde{E}_k(l_1, \ldots, l_k) \, M_\gamma^k) \qquad (3.2.37)$$

and from the definition of M_γ in (3.2.10) it is easy to check that there is a constant c such that

$$M_\gamma^k \leqslant \exp(ck[\log k + \log |\Lambda_{q+1}| + \log(\gamma + 3)]) \qquad (3.2.38)$$

From (3.2.12) and (3.2.10) we also have that for some $c > 0$

$$\begin{aligned}
&\mathrm{card}(\tilde{E}_k(l_1, \ldots, l_k)) \\
&\quad \leqslant \exp[ck(\log k + \log |\Lambda_{q+1}|] \prod_{i=1}^{k} \exp[c(l_i + 2)] \qquad (3.2.39)
\end{aligned}$$

We combine the definition of $C_{1,2}$ in (3.2.17), and (3.2.37)–(3.2.39) to

(3.2.36) and we use the fact that $k \log \gamma \leqslant k \log k + \gamma \log \gamma$ for $\gamma > 1$ to conclude that there are constants c_1 and c_2 such that

$$M(q, k); l_1, ..., l_k) \leqslant c_1^k (n-k)! \left[\prod_{i=1}^{k} \exp\left(-\frac{l_i^2}{32\beta} \right) \right] \times \exp[c_2 k(\log k + \log |\Lambda_{q+1}|)] \qquad (3.2.40)$$

We substitute (3.2.40) into (3.2.35) and (3.2.35) into (3.2.8) to obtain the following bound:

$$\begin{aligned} \bar{F}^{(n)}(q, k) &\leqslant c_1^k \exp[c_2 k(\log k + \log |\Lambda_{q+1}|)] \\ &\quad \times \operatorname{Tr}_{\mathscr{H}^{(s)}_{n-k}(\Lambda)}(\exp[-\beta H_\Lambda^{(n-k)}]) \\ &\quad \times \prod_{i=1}^{k} \left[\sum_{\Delta_i \subset \Lambda_{q+1}} \sum_{l_i = -1}^{\infty} \exp\left[-\frac{1}{32\beta} l_i^2 + 4\nu b(l_i + 2)^2 \right] \right] \\ &\leqslant d_i^k \exp[d_2(k \log k + |\Lambda_{q+1}| \log |\Lambda_{q+1}|] \operatorname{Tr}_{\mathscr{H}^{(s)}_{n-k}}(\exp[-\beta H_\Lambda^{(n-k)}]) \end{aligned} \qquad (3.2.41)$$

Here we have choose b such that $4\nu b < 1/32\beta$ and we have used the fact that $k(\log k + \log \gamma) \leqslant \text{const}(k \log k + \gamma \log \gamma)$ to get the second inequality. Next we note that for any $\varepsilon > 0$ and for any $c_1, c_2, c_3, c > 0$ there is a constant $d_1 > 0$ such that

$$c_1 k + c_2(k \log k + |\Lambda_{q+1}| \log |\Lambda_{q+1}|) - c(k^{1+\varepsilon} + |\Lambda_q|^{1+\varepsilon}) \leqslant d_1 \qquad (3.2.42)$$

We finally combine (3.2.41) and (3.2.1) and then use (3.2.42) to obtain

$$\bar{G}^{(n)}(q, k) \leqslant \exp[-c' |\Lambda_{q+1}|^{1 + [(l-1)(1+\varepsilon)/P]} - c' k^{1+\varepsilon} + d_1] \times \operatorname{Tr}_{\mathscr{H}^{(s)}_{n-k}(\Lambda)}(\exp[-\beta H_\Lambda^{(n-k)}]$$

for some constant c', $d_1 > 0$. We substitute the above bound to (3.1.24) to conclude that

$$\sum_{n=0}^{\infty} \sum_{q \geqslant q_0} \sum_{k} G^{(n)}(q, k) \leqslant e^D \Xi_\Lambda^{-1} \sum_{n'=0}^{\infty} z^{n'} \operatorname{Tr}_{\mathscr{H}^{(s)}_{n'}}(\exp[-\beta H_\Lambda^{(n')}]) = e^D$$

for some $D > 0$. Proposition 2.2.1 follows from (3.1.20), (3.1.24), and the above bound. This complete the proof of Proposition 2.2.1. ∎

4. CONTROL OF INTERACTIONS: PROOF OF THEOREM 3.1.1

In this section we complete the proof of Proposition 2.2.1 by proving Theorem 3.1.1. We start with some technical estimates. Let $F_q(p)$ and $\gamma(v, 2)$ be defined as in (3.1.12) and (3.1.10), respectively. The following result corresponds to Proposition IV.4.1 and Proposition IV.4.3 of Ref. 10.

Lemma 4.1. For a given path configuration $(\omega)_n \in \bar{\mathscr{E}}_{q,k}$, let us assume that

$$d_1 F_q(p) \geqslant |A_q|^l \qquad \text{for some} \quad d_1 > 0$$

and let $1 < l < p < \gamma(v, 2)$. Then for any given $a_1 > 0$ and $b_1 > 0$ there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$a_1 U_1((\omega)_k) + b_1 V((\omega)_k) \geqslant c_1 |A_{q+1}|^{1+(l-1)\gamma(v,2)/p} - c_2 k$$

on $\bar{\mathscr{E}}_{q,k}$.

Proof. By the assumption in the lemma there exists $\tau_0 \in [0, \beta]$ such that

$$d_1 \sum_{Q(\gamma) \subset A_q} n(\gamma, \tau_0)^p \geqslant |A_q|^l$$

We fix $\tau_0 \in [0, \beta]$ so that the above inequality holds. Let

$$B(\tau) \equiv a_1 U_1((\omega(\tau))_k) + b_1 V((\omega(\tau))_k) \tag{4.1}$$

where

$$V((\omega(\tau))_k) = \frac{1}{\beta} \sum_{i=1}^{k} |\omega_i(\tau) - \omega_i(\tau_0)|^2 \tag{4.2}$$

and $U_1((\omega(\tau))_k)$ has been defined in (3.1.28). We denote that $\tilde{n}(r, \tau)$ is the number of ω_i, $i = 1,..., k$ such that $\omega_i(\tau) \in Q(r)$, $\bar{\omega}_l$ be the path hitting A_q at $\tau = \tau_0$ [i.e., $\bar{\omega}_l(\tau_0) \in A_q$], and $\bar{n}(r, \tau)$ be the number of $\bar{\omega}_l$'s such that $\bar{\omega}_l(\tau) \in Q(r)$. By the definition we have

$$d_1 \sum_{Q(r) \subset A_q} \bar{n}(r, \tau_0)^p \geqslant |A_q|^l \tag{4.3}$$

and by the superstability condition [Assumption A(a)]

$$\begin{aligned} B(\tau) &\geqslant a_1 \sum_{Q(r) \subset A} [A\tilde{n}(r, \tau)^2 - B\tilde{n}(r, \tau)] + b_1 V((\omega)_k) \\ &\geqslant \sum_{Q(r) \subset A} \Bigg[a_1 \bar{n}(r, \tau)^2 + \frac{b_1}{\beta} \\ &\qquad \times \sum_{\bar{\omega}_l(\tau) \in Q(r)} |\bar{\omega}_l(\tau) - \bar{\omega}_l(\tau_0)|^2 \Bigg] - a_1 Bk \end{aligned} \tag{4.4}$$

We now employ the method used in the proof of Proposition IV.4.1 of Ref. 10. Let $\bar{n}_{r'}(r, \tau)$ be the number of $\bar{\omega}_i$'s such that $\bar{\omega}_i(\tau) \in Q(r)$ and $\bar{\omega}_i(\tau_0) \in Q(r')$. Then

$$\bar{n}(r, \tau)^2 \geqslant \sum_{Q(r') \subset \Lambda_q} \bar{n}_{r'}(r, \tau)^2$$

and so

$$B(\tau) \geqslant \sum_{Q(r') \subset \Lambda_q} \left[a_1 A \sum_{Q(r) \subset \Lambda} \bar{n}_{r'}(r, \tau)^2 + \frac{b_1}{\beta} \sum_{\substack{\bar{\omega}_i(\tau) \in Q(r): \\ \bar{\omega}_i(\tau_0) \in Q(r')}} |\bar{\omega}_i(\tau) - \bar{\omega}_i(\tau_0)|^2 \right] - a_1 Bk$$

We have arrived at the expression which is the same as that in (4.4.12) of Ref. 10.

Consider the case in which for a given $q > 0$

$$\operatorname{card}(\{\bar{\omega}_i : \bar{\omega}_i(\tau_0) \in Q(r') \text{ and } |\bar{\omega}_i(\tau) - \bar{\omega}_i(\tau_0)| \leqslant \bar{n}(r', \tau_0)^q\}) \geqslant [\tfrac{1}{2}\bar{n}(r', \tau_0)]$$

Since the number of $Q(r)$'s such that $|r - r'| \leqslant \bar{n}(r', \tau_0)^q$ is not larger than $[2\bar{n}(r', \tau_0)^q + 1]^\nu$ we use Hölder's inequality to obtain

$$\sum_{Q(r) \subset \Lambda} \bar{n}_{r'}(r, \tau)^2 \geqslant c_1' \bar{n}(r', \tau_0)^{\nu q} \, \bar{n}(r', \tau_0)^{(1 - \nu q)^2} - c_2'$$

for some constant c_1' and c_2'. On the other hand if

$$\operatorname{card}(\{\bar{\omega}_i : \bar{\omega}_i(\tau_0) \in Q(r') \text{ and } |\bar{\omega}_i(\tau) - \bar{\omega}_i(\tau_0)| \leqslant \bar{n}(r', \tau_0)^q\}) < [\tfrac{1}{2}\bar{n}(r', \tau_0)]$$

then

$$\sum_{\bar{\omega}_i(\tau) \in Q(r) : \bar{\omega}_i(\tau_0) \in Q(r')} |\bar{\omega}(\tau) - \bar{\omega}_i(\tau_0)|^2 \geqslant c_1' \bar{n}(r', \tau_0)^{1 + 2q} - c_2'$$

for some constant c_1' and c_2'. Choosing q such that $\nu q + (1 - \nu q)\, 2 = 1 + 2q$ (i.e., $q = 1/[2\nu + (2 - \nu)]$) we obtain

$$B(\tau) \geqslant c \sum_{Q(r) \subset \Lambda_q} \bar{n}(r, \tau_0)^{\gamma(\nu, 2)} - a_1 Bk - c_2 |\Lambda_q| \tag{4.5}$$

For the detailed derivation of (4.5) from (4.4) we refer to the proof of Proposition IV.4.1 of Ref. 10.

We note that

$$a_1 U_1((\omega)_k) + b_1 V((\omega)_k) \geqslant \int_0^\beta B(\tau)\, d\tau \tag{4.6}$$

We use Hölder's inequality to obtain

$$\left[|A_q|^{-1} \sum_{Q(r) \subset A_q} \bar{n}(r, \tau_0)^\gamma \right]^{p/\gamma} \geqslant |A_q|^{-1} \sum_{Q(r) \subset A_q} \bar{n}(r, \tau_0)^p$$

and so by (4.3)

$$\sum_{Q(r) \subset A_q} \bar{n}(r, \tau_0)^\gamma \geqslant |A_q|^{1 + (l-1)\gamma/p} \tag{4.7}$$

The lemma now follows from (4.5), (4.6), and (4.7). ∎

Proposition 4.2. For any $a_1 > 0$ and $b_1 > 0$ one can choose q_0 large enough such that for $1 < l < p < \gamma(\nu, 2)$ there exist constant $c_1 > 0$ and $c_2 > 0$ such that

$$a_1 U((\omega)_k) + b_1 V((\omega)_k) \geqslant c_1 |A_{q+1}|^{1 + (l-1)(1+\varepsilon)/p} - c_2 k$$

on $\bar{\mathscr{E}}_{q,k}$.

Proof. If $\frac{1}{2} F_q(p) \geqslant |A_q|^l$, the proposition follows from Lemma 4.1 and the fact that $1 < \gamma(\nu, 2)$. On the other hand if $\frac{1}{2} F_q(p) < |A_q|^l$ on $\bar{\mathscr{E}}_{q,k} \subset \mathscr{E}_q$, we have that

$$E_q = \sum_{Q(r) \subset A_a} \int_0^\beta n(r, \tau)^2\, d\tau \geqslant \tfrac{1}{2}\, |A_q|^l$$

by the definition of $\mathscr{E}_q$ in (3.1.14). Thus by Assumption A(a)

$$\begin{aligned} a_1 U((\omega)_k) &\geqslant a_1 \sum_{Q(r) \subset A} \int_0^\beta d\tau [An(r, \tau)^2 - Bn(r, \tau)] \\ &\geqslant c_1\, |A_q|^l - c_2 k \end{aligned} \tag{4.8}$$

on $\mathscr{E}_{q,k}$ for some constant c_1 and c_2. We choose l and p such that $1 < l < p < \gamma(\nu, 2)$. Then $1 + [(l-1)/p] < l$ and so the proposition follows from (4.8) and and the definition of A_q in (3.1.9). This completes the proof of the proposition. ∎

Lemma 4.3. For any $a_2 >$ and $b_2 > 0$ there exist $c_3 > 0$ and $\varepsilon > 0$ such that for sufficiently large q_0, $1 < l < p < \gamma(v, 2)$, and for sufficiently small α

$$a_2 U_1((\omega)_k) + b_2 V((\omega)_k) \geqslant c_3 k^{1+\varepsilon}$$

on $\bar{\mathscr{E}}_{q,k}$.

Proof. If $k \leqslant c\,|\Lambda_q|^{1+(l-1)/p}$ for some constant, the lemma follows from Proposition 4.2. Thus we next consider the case in which

$$k \geqslant c\,|\Lambda_q|^{1+(l-1)/p}$$

for some $c > 0$. Let

$$k' = \text{card}(\{\omega_i : \omega_i(\tau) \in \Lambda_{q+\log k} \text{ for all } \tau \in [0, \beta],\ i = 1,..., k\}$$

We consider the case in which $\frac{1}{2}k \leqslant k'$. Let $\tilde{n}(r, \tau)$ be the number of $\omega_i \in \{\omega_1,..., \omega_k\}$ such that $\omega_i(\tau) \in Q(r)$. Then by the superstability

$$\begin{aligned} a_2 U_1((\omega)_k) &\geqslant a_2 \int_0^\beta \sum_{Q(r) \subset \Lambda} [A\tilde{n}(r, \tau)^2 - B\tilde{n}(r, \tau)]\, d\tau \\ &\geqslant a_2 \int_0^\beta \left[A \sum_{Q(r) \subset \Lambda_{q+\log k}} \tilde{n}(r, \tau)^2 \right] d\tau - c'k \end{aligned} \tag{4.9}$$

for some constant c'. We use Hölder's inequality to obtain

$$\begin{aligned} \sum_{Q(r) \subset \Lambda_{q+\log k}} \tilde{n}(r, \tau)^2 &\geqslant |\Lambda_{q+\log k}| \left[|\Lambda_{q+\log k}|^{-1} \sum_{Q(r) \subset \Lambda_{q+\log k}} \tilde{n}(r, \tau) \right]^2 \\ &\geqslant \tfrac{1}{4}k^2\, |\Lambda_{q+\log k}|^{-1} \end{aligned}$$

if $k' \geqslant \frac{1}{2}k$. Thus from (4.9) and the definition of Λ_q it follows that

$$a_2 U_1((\omega)_k) \geqslant c''\, |\Lambda_q|^{-1}\, k^{-\alpha v} k^2 - c'k \tag{4.10}$$

For small α, the above is bigger than $k^{1+\varepsilon}$ for some $\varepsilon > 0$. Next we consider the case in which $k' < \frac{1}{2}k$. This means that more than $\frac{1}{2}k$ paths hit Λ_{q+1} and $\Lambda^c_{q+\log k}$ during the time interval $[0, \beta]$. Thus

$$\begin{aligned} b_2 V((\omega)_k) &\geqslant c\, \tfrac{1}{2}k[l_{q+\log k} - l_{q+1}]^2 \\ &\geqslant c'k[e^{\alpha \log(k-1)} - 1]\, l^2_{q+1} \\ &\geqslant c''k^{l+\varepsilon} \end{aligned} \tag{4.11}$$

Thus in either case we have the bound

$$a_2 U_1((\omega)_k) + b_2 V((\omega)_k) \geqslant ck^{1+\varepsilon}$$

This proves the lemma. ∎

Corollary 4.4. For any $a_1 > 0$ and $b_1 > 0$ one can choose α small enough, q_0 large enough such that for $1 < l < p < \gamma(v, 2)$ there exist constant $c_1 > 0$ and $\varepsilon > 0$ such that

$$a_1 U_1((\omega)_k) + b_1 V((\omega)_k) \geqslant c_1 [\,|\Lambda_{q+1}|^{1+(l-1)(1+\varepsilon)/p} + k^{1+\varepsilon}]$$

on $\bar{\mathscr{E}}_{q,k}$

Proof. The above is a consequence of Proposition 4.2 and Lemma 4.3. ∎

Proposition 4.5. For any $b_2 > 0$, one can choose α small enough and q_0 large enough such that for $1 < l < p < \gamma(v, 2)$

$$U_1((\omega)_k) + b_2 V((\omega)_k) \geqslant \tfrac{9}{10} A\, |\Lambda_q|^l$$

on $\bar{\mathscr{E}}_{q,k}$, where A is the constant in Assumption A(a).

Proof. If $20F_q(p) \geqslant |\Lambda_q|^l$, we choose p such that $l < 1 + (l-1)\,\gamma(v, 2)/p$ $(1 < \gamma(v, 2)/p)$. We then apply Lemma 4.1 and Corollary 4.4 to get the bound in the proposition. On the orther hand, if $20F_q(p) < |\Lambda_q|^l$, then $E_q \geqslant \frac{19}{20} |\Lambda_q|^l$. Then it follows that

$$U_1((\omega)_k) \geqslant \tfrac{19}{20} A\, |\Lambda_q|^l - B\beta k$$

and so the bound in the proposition follows from the above bound and Corollary 4.4. This proves the proposition. ∎

We next control the $W((\omega)_k, (\omega)_{n-k})$ term. We will employ the method similar to that used in Ref. 2. For a given path configuration $(\omega)_n$ in $\bar{\mathscr{E}}_{q,k}$ we divide $(\omega)_k = \{\omega_1, w_2,..., \omega_k\}$ into the following two classes:

$$(\omega)_k = (\zeta) \cup (\xi) \tag{4.12}$$

where

$$\begin{aligned} &(\zeta)\text{:} \quad \text{the paths completely contained in } \Lambda_{q+3} \\ &(\xi)\text{:} \quad \text{the paths which cross the boundary } \partial\Lambda_{q+3} \text{ of } \Lambda_{q+3} \end{aligned} \tag{4.13}$$

The following results correspond to Lemma 2.2 and Lemma 2.3 of Ref. 2, respectively.

Lemma 4.6. If $(\tilde{\omega})$ and $(\hat{\omega})$ are sets of paths of $(\omega)_n$ contained in

$\Lambda_{q+1+a}(a \geqslant 0)$ and Λ^c_{q+1}, respectively, then there exist α small enough and q_0 large enough such that for $q_0 \leqslant q$

$$W((\tilde{\omega}), (\hat{\omega})) \geqslant -\tfrac{1}{16} A \, |\Lambda_{q+1}|^l$$

on $\mathscr{E}_q$, where A is the constant in Assumption A(a).

Lemma 4.7. Let (ζ) be the path contained in Λ_s, $s > q+1$. Then there is a q_0 large enough such that for each $q \geqslant q_0$

$$W((\zeta), (\omega)_{n-k}) \geqslant -d_1 \, |\zeta| \, |\Lambda_s|^{l/2}$$

on $\mathscr{E}_q$ for some constants $d_1 d_2 > 0$, where $|\zeta|$ is card$((\zeta))$.

We leave the proofs of Lemma 4.6 and Lemma 4.7 to the end of this section. Using the above lemmas we now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. From Corollary 4.4, Proposition 4.5, and Lemma 4.6 it follows that

$$\begin{aligned}
&U((\omega)_k) + bV((\omega)_k) + W((\omega)_k, (\omega)_{n-k}) \\
&\quad = \frac{9}{10} U_1((\omega)_k) + \frac{1}{10} U_1((\omega)_k) + \frac{b}{2} V((\omega)_k) + W((\zeta), (\omega)_{n-k}) \\
&\qquad + \frac{b}{2} V((\omega)_k) + W((\xi), (\omega)_{n-k}) \\
&\quad \geqslant \left(\frac{9}{10}\right)^2 A \, |\Lambda_q|^l + c'[|\Lambda_{q+1}|^{1+(l-1)(1+\varepsilon)/p} + k^{1+\varepsilon}] - \frac{1}{16} A \, |\Lambda_{q+1}|^l \\
&\qquad + \frac{b}{2} V((\omega)_k) + W((\xi), (\omega)_{n-k})
\end{aligned} \tag{4.14}$$

Note that, if the interaction is repulsive, $W((\xi), (\omega)_{n-k}) \geqslant 0$. In this case the theorem follows from the above inequality. We consider the general case. Let $(\xi^{(s)}) \subset (\omega)_k$ be the paths which are contained in Λ_s and cross the boundary $\partial\Lambda_{s-1}$ of Λ_{s-1}. Obviously we have that by Lemma 4.7

$$\begin{aligned}
&\frac{b}{2} V((\omega)_k) + W((\xi), (\omega)_{n-k}) \\
&\quad \geqslant \sum_{s=q+4}^{\infty} \left[\frac{b}{2} V((\xi^{(s)})) + W((\xi^{(s)}), (\omega)_{n-k})\right] \\
&\quad \geqslant \sum_{s=q+4}^{\infty} \left[\frac{b}{2} |\xi^{(s)}| \, (l_s - l_{q+1})^2 - d \, |\xi^{(s)}| \, |\Lambda_s|^{l/2}\right] \\
&\quad \geqslant \sum_{s=q+4}^{\infty} |\xi^{(s)}| \left[\frac{b}{2} (l_s - l_{q+1})^2 - 3^v d l_s^{vl/2}\right]
\end{aligned}$$

For $v \leqslant 3$ we choose $l > 1$ such that $\frac{1}{2}vl < 2$ [and also $1 < \gamma(v, 2)/p$ so that Proposition 4.5 holds]. Notice that $l_s = [e^{\alpha s}]$. The expressions in the bracket in the last inequality become positive for sufficiently large $q_0 (q_0 \leqslant q)$. Thus for sufficiently large q_0 we have

$$\frac{b}{2} V((\omega)_k) + W((\xi), (\omega)_{n-k}) \geqslant 0 \tag{4.15}$$

Thus for small α the theorem follows from (4.14) and (4.15). This completes the proof of Theorem 3.1.1. ▌

In the rest of this section we prove Lemma 4.6 and Lemma 4.7. We first prove Lemma 4.7.

Proof of Lemma 4.7. We use the method similar to that used in the proof of Lemma 2.3 of Ref. 2. By the lower regularity in Assumption A(b)

$$\begin{aligned} &-W((\zeta(\tau)), (\omega(\tau))_{n-k}) \\ &\quad \leqslant \sum_{Q(r) \subset \Lambda_s} \sum_{Q(r') \subset \Lambda^c_{q+1}} \psi(r, r')\, n(r, (\zeta(\tau)))\, n(r', (\omega(\tau))) \end{aligned}$$

where $n(r, (\zeta(\tau)))$ is the number of the paths $\zeta_i \in (\zeta)$ with $\zeta_i(\tau) \in Q(r)$, and $n(r', (\omega(\tau)))$ is the number of the paths $\omega_l \in (\omega)_{n-k}$ with $\omega_l(\tau) \in Q(r')$. We use Hölder's inequality to obtain

$$\begin{aligned} &-W((\zeta(\tau)), (\omega(\tau))_{n-k}) \\ &\quad \leqslant \sum_{k=0}^{\infty} \psi(k) \sum_{Q(r) \subset \Lambda_s} n(r, (\zeta(\tau))) \sum_{\substack{Q(r') \subset \Lambda^c_{q+1} \\ d(r,r') = k}} n(r', (\omega(\tau))) \\ &\quad \leqslant v_1 \sum_{k=0}^{\infty} \psi(k)(k+1)^{(v-1)/2} \sum_{Q(r) \subset \Lambda_s} n(r, (\zeta(\tau))) \\ &\qquad \times \left[\sum_{\substack{Q(r') \subset \Lambda^c_{q+1} \\ d(r,r') = k}} n(r', (\omega(\tau)))^2 \right]^{1/2} \end{aligned} \tag{4.16}$$

for some constant v_1 depending on v. But

$$\sup_{Q(r) \subset \Lambda_s} \sum_{\substack{Q(r') \subset \Lambda^c_{q+1} \\ d(r,r') = k}} n(r', (\omega(\tau)))^2 \leqslant \sum_{Q(r') \subset \Lambda_s + r(k)} n(r', (\omega(\tau)))^2 \tag{4.17}$$

where $r(k)$ is the smallest of integers r such that the set $\{Q(r') : d(r, r') = k$ for all $Q(r) \subset \Lambda_s\}$ is contained in Λ_{s+r}. Thus we substitute (4.17) into (4.16). By integrating

$$-W((\zeta), (\omega)_{n-k}) \leqslant v_1 |\zeta| \sum_{k=0}^{\infty} \psi(k)(k+1)^{(v-1)/2} |\Lambda_{s+r(k)}|^{l/2} \tag{4.18}$$

on $\mathscr{E}_q$. By Lemma B.2 of Ref. 2

$$r(k) \leqslant 1 + \frac{1}{\alpha} \log(k+2) \tag{4.19}$$

We also note that by (4.19) and the definition of Λ_q

$$\begin{aligned} \left(\frac{|\Lambda_{s+r(k)}|}{|\Lambda_s|} \right)^{l/2} &\leqslant \exp[\alpha v l r(k)] \\ &\leqslant c(k+1)^{vl/2} \end{aligned} \tag{4.20}$$

for some constant depending on v. From (4.18) and (4.20) it follows that

$$-W((\zeta), (\omega)_{n-k}) \leqslant c' \, |\zeta| \, |\Lambda_s|^{l/2} \sum_{k=0}^{\infty} \psi(k)(k+1)^{(vl+v-1)/2}$$

If we choose $l = 1 + \delta$ for sufficiently small δ, then lemma follows from the lower regularity (2.2.3) and the above bound. ∎

We finally prove Lemma 4.6 by using the method similar to that used in the proof of Lemma 2.2. of Ref. 2, and so we complete the proof of Theorem 3.1.1.

Proof of Lemma 4.6. We note that by Assumption A(b)

$$\begin{aligned} &-W((\tilde{\omega}(\tau)), (\hat{\omega}(\tau))) \\ &\leqslant \frac{1}{2} \sum_{Q(r) \subset \Lambda_{q+1+a}} \sum_{Q(r') \subset \Lambda^c_{q+1}} \psi(r, r')[n(r, (\omega(\tau)))^2 + n(r', (\omega(\tau)))^2] \\ &= T_1 + T_2 + T_3 + T_4 + T_5 \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} T_1 &= \frac{1}{2} \sum_{Q(r) \subset \Lambda_{q+1+a} \backslash \Lambda_q} \sum_{Q(r') \subset \Lambda_{q+2+a} \backslash \Lambda_{q+1}} \psi(r, r') \, n(r)^2 \\ T_2 &= \frac{1}{2} \sum_{Q(r) \subset \Lambda_q} \sum_{Q(r') \subset \Lambda_{q+2+a} \backslash \Lambda_{q+1}} \psi(r, r') \, n(r)^2 \\ T_3 &= \frac{1}{2} \sum_{Q(r) \subset \Lambda_{q+1+a}} \sum_{Q(r') \subset \Lambda^c_{q+2+a}} \psi(r, r') \, n(r)^2 \\ T_4 &= \frac{1}{2} \sum_{Q(r) \subset \Lambda_{q+1+a}} \sum_{Q(r') \subset \Lambda_{q+2+a} \backslash \Lambda_{q+1}} \psi(r, r') \, n(r')^2 \\ T_5 &= \frac{1}{2} \sum_{Q(r) \subset \Lambda_{q+1+a}} \sum_{Q(r') \subset \Lambda^c_{q+2+a}} \psi(r, r') \, n(r')^2 \end{aligned} \tag{4.22}$$

Here $n(r)=n(r, (\tilde{\omega}(\tau)))$ and $n(r')=n(r', (\hat{\omega}(\tau)))$. Let

$$
\begin{aligned}
F &\equiv \sum_{k=0}^{\infty} \psi(k)(k+1)^{\nu-1} < \infty \\
F(l) &\equiv \sum_{k=l}^{\infty} \psi(k)(k+1)^{\nu-1} \\
\psi_k &\equiv \sum_{k'=l_{q+1+a+k}-l_{q+1+a}} \psi(k')
\end{aligned}
\tag{4.23}
$$

Then the following are easy to obtained:

$$
\begin{aligned}
T_1 &\leqslant \frac{1}{2} F \sum_{Q(r)\subset \Lambda_{q+1+a}\setminus\Lambda_q} n(r)^2 \\
T_2 &\leqslant \frac{1}{2} F(l_{q+1}-l_q) \sum_{Q(r)\subset\Lambda_q} n(r)^2 \\
T_3 &\leqslant \frac{1}{2} F(l_{q+2+a}-l_{q+1+a}) \sum_{Q(r)\subset\Lambda_q+1+a} n(r)^2 \\
T_4 &\leqslant \frac{1}{2} F \sum_{Q(r')\subset\Lambda_{q+2+a}\setminus\Lambda_{q+1}} n(r')^2 \\
T_5 &\leqslant \frac{1}{2} \sum_{Q(r)\subset\Lambda_{q+1+a}} \sum_{k=1}^{\infty} \psi_k \left[\sum_{Q(r')\subset\Lambda_{q+2+a+k}\setminus\Lambda_{q+1+a+k}} n(r')^2\right]
\end{aligned}
\tag{4.24}
$$

By integrating with respect to τ and by noting that $l_{q+1}-l_q \leqslant l_{q+a+2}-l_{q+a+1}$, we obtain that

$$
\begin{aligned}
-W((\tilde{\omega}),(\hat{\omega})) &\leqslant FE_{\Lambda_{q+2+a}\setminus\Lambda_q} + F(l_{q+1}-l_q)\, E_{\Lambda_{q+1+a}} \\
&\quad + \frac{1}{2} \sum_{Q(r)\subset\Lambda_{q+1+a}} \sum_{k=1}^{\infty} \psi_k [E_{\Lambda_{q+2+a+k}\setminus\Lambda_{q+a+2}} - E_{\Lambda_{q+1+a+k}\setminus\Lambda_{q+2+a}}] \\
&\leqslant FE_{\Lambda_{q+2+a}\setminus\Lambda_q} + F(l_{q+1}-l_q)\, E_{\Lambda+1+a} \\
&\quad + \frac{1}{2} \sum_{Q(r)\subset\Lambda_{q+1+a}} \sum_{k=1}^{\infty} (\psi_k - \psi_{k+1})\, E_{\Lambda_{q+2+k+a}}
\end{aligned}
\tag{4.25}
$$

Here for $B\subset\mathbf{R}^\nu$ we have written

$$
E_B \equiv \sum_{Q(r)\subset B} \int_0^\beta n(r,\tau)^2\, d\tau \tag{4.26}
$$

We assert that, if q_0 is sufficiently large ($q_0 \leqslant q$) and if $l = 1 + \delta$ with sufficiently small $\delta > 0$, then the following bounds hold:

$$\begin{gathered}
[|\Lambda_{q+2+a}|^l - |\Lambda_q|^l]/|\Lambda_{q+1}|^l < A/48F \\
[|\Lambda_{q+1+a}|^l F(l_{q+1} - l_q)]/|\Lambda_{q+1}|^l < A/48 \\
|\Lambda_{q+1+a}|\ |\Lambda_{q+1}|^{-l} \sum_{k=1}^{\infty} (\psi_k - \psi_{k+1})\ |\Lambda_{q+2+a+k}|^l < A/24
\end{gathered} \tag{4.27}$$

for α small enough. We prove our assertions. By the definition of Λ_q in (4.1.9) the left expression in the first inequality is bounded by $\exp[\alpha\nu(a+1)] - 1$, and so the first inequality holds for sufficiently small α. We note that $l_{q+1} - l_q > \alpha l_q$ and so $F(l_{q+1} - l_q) \to 0$ as $q \to \infty$. This proves the second inequality. To prove the last inequality in (4.27) we note that

$$\begin{aligned}
&|\Lambda_{q+2+a+k}|^l/[2(l_{q+2+a+k} - l_{q+2+a}) + 3]^{\nu l} \\
&\quad\leqslant [l_{q+a+2+k}/(l_{q+1+a+k} - l_{q+a+1})]^{\nu l} \\
&\quad\leqslant [e^{\alpha}/(1 - e^{-\alpha k})]^{\nu l} \\
&\quad\leqslant [e^{\alpha}/(1 - e^{-\alpha})]^{\nu l}
\end{aligned}$$

Thus the left side of the third inequality in (4.29) is bounded by

$$\begin{aligned}
&[e^{\alpha}/(1 - e^{-\alpha})]^{\nu l} e^{\alpha\nu a} \sum_{k=1}^{\infty} (\psi_k - \psi_{k+1})[2(l_{q+2+a+k} - l_{q+a+1})]^{\nu l} \\
&\quad\leqslant [e^{\alpha}/(1 - e^{-\alpha})]^{\nu l} e^{\alpha\nu a} \sum_{s = l_{q+2+a} - l_{q+1+a}} [\psi(s) - \psi(s+1)][2(s+3)]^{\nu l} \\
&\quad\to 0 \qquad \text{as} \quad q \to \infty
\end{aligned}$$

by the lower reguarity and the fact that $l_{q+2+a} - l_{q+1+a} \to \infty$ as $q \to \infty$. This proves the third inequality in (4.27).

We note that for $q_0 \leqslant q < q'$ that on $\mathscr{E}_q$

$$\begin{aligned}
E_{\Lambda_{q'}} &< |\Lambda_{q'}|^l \\
E_{\Lambda_{q'} \setminus \Lambda_q} &< E_{\Lambda_{q'}} - E_{\Lambda_q} \\
&\leqslant (E_{\Lambda_{q'}} - E_{\Lambda_q}) + [F_{q'}(p) - F_q(p)] \\
&< |\Lambda_{q'}|^l - |\Lambda_q|^l
\end{aligned} \tag{4.28}$$

where for $B \subset \mathbf{R}^{\nu}$

$$F_B(p) \equiv \sup_{\tau \in [0, \beta]} \sum_{Q(r) \subset B} n(r, \tau)^p$$

Thus the lemma follows from (4.25), (4.27), and (4.28). This completes the proof of Lemma 4.6. ▌

5. PRESSURE AND EQUIVALENCE OF THE BOUNDARY CONDITIONS: PROOF OF THEOREM 2.2.4

We will prove Theorem 2.2.4 in the following manner. From (2.2.12) it follows that

$$P_\Lambda = P_{\infty,\Lambda} \leqslant P_{\sigma,\Lambda} \leqslant P_{0,\Lambda} \tag{5.1}$$

Thus, if one can prove that the limit

$$P = \lim_{\Lambda \uparrow \mathbf{R}^\nu} P_\Lambda \tag{5.2}$$

exists as Λ tends to $\mathbf{R}^\nu$ in the sense of van Hove, and that

$$P_{0,\Lambda} - P_\Lambda \to 0 \qquad \text{as} \quad \Lambda \to \mathbf{R}^\nu \tag{5.3}$$

one proves the existence of the infinite volume limit pressure and its independence of the boundary conditions.

We begin to prove Theorem 2.2.4 by showing the existence of the pressure with Dirichlet boundary conditions.

Proposition 5.1. Let Λ tend to $\mathbf{R}^\nu$ in the sense of van Hove. Under the assumptions in Theorem 2.2.4 the limit

$$P = \lim_{\Lambda \uparrow \mathbf{R}^\nu} P_\Lambda$$

exists.

Proof. Let us consider the following rectangular regions:

$$\begin{aligned} \Lambda &= (-l_1, l_1) \times S \\ \Lambda_1 &= (-l_1, -\tfrac{1}{2}b) \times S \\ \Lambda_2 &= (\tfrac{1}{2}b, l_1) \times S \\ S &= \prod_{i=2}^{\nu} (-l_i, l_i) \end{aligned} \tag{5.4}$$

where $b > 0$ is the constant given in Theorem 2.2.4. We will show that

$$\exp[-m_0 |S|]\, \Xi_{\Lambda_1} \Xi_{\Lambda_2} \leqslant \Xi_\Lambda \tag{5.5}$$

for some constant $m_0 \geqslant 0$. The proposition then follows for translation-invariant interactions by a standard argument in statistical mechanics.[1,13]

We prove (5.5). Since $\Lambda_1 \cup \Lambda_2 \subset \Lambda$, it follows that

$$\Xi_\Lambda \geqslant \mathrm{Tr}_{\mathrm{F}^{(s)}(\Lambda_1 \cup \Lambda_2)}\{\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2} + W(\Lambda_1, \Lambda_2))]\} \tag{5.6}$$

where $W(\Lambda_1, \Lambda_2)$ is the operator corresponding to the interaction between the particles in Λ_1 and the particles in Λ_2. The decay property in Theorem 2.2.4 yields

$$W(\Lambda_1, \Lambda_2) \leqslant \frac{1}{2} \sum_{\substack{Q(r) \subset \Lambda_1 \\ Q(r') \subset \Lambda_2}} \tilde{\varphi}(|r - r'|)[N_{Q(r)}^2 + N_{Q(r')}^2] \tag{5.7}$$

We use the Peierls–Bogoliubov-type inequality [see (2.16) of Ref. 9]

$$\mathrm{Tr}(e^{A+B}) \geqslant \mathrm{Tr}(e^A) \exp\{\mathrm{Tr}(Be^A)/\mathrm{Tr}(e^A)\} \tag{5.8}$$

for trace class operators e^A and e^{A+B}. Applying (5.8) and (5.7) to (5.6) and using the bounds in (2.2.10) we obtain

$$\Xi_\Lambda \geqslant \Xi_{\Lambda_1} \Xi_{\Lambda_2} \exp\left\{-c \sum_{\substack{Q(r) \subset \Lambda_1 \\ Q(r') \subset \Lambda_2}} \tilde{\varphi}(|r - r'|)\right\}$$

The bounds in (5.5) now follows from the decay property in (2.2.14) and the above bounds. This completes the proof of the proposition. ∎

We next prove (5.3) and so complete the proof of Theorem 2.2.4.

Proposition 5.2. Let the interaction satisfy Assumption A. Then

$$P_{0,\Lambda} - P_\Lambda \to 0$$

as Λ tends to $\mathbf{R}^\nu$ in the sense of van Hove.

The main idea in proving the above proposition is careful applications of the methods in Section 3 and Section 4 with convex combinations of Wiener measures corresponding to different boundary conditions.

Before proving the proposition we need some preparations. Let $P^\beta_{\sigma,\Lambda}(x, y; d\omega)$ be the conditional Wiener measure corresponding to the transition function

$$P^t_{\sigma,\Lambda}(x, y) = \exp[-\tfrac{1}{2}t\,\Delta_{\sigma,\Lambda}](x, y)$$

the kernel of $\exp[-\frac{1}{2}t\,\Delta_{\sigma,\Lambda}]$. For a notational convenience we write that for $s \in [0, 1]$

$$P_{\Lambda,s}(x, y; d\omega) = sP^\beta_{0,\Lambda}(x, y; d\omega) + (1 - s)\, P^\beta_{\infty,\Lambda}(x, y; d\omega) \tag{5.9}$$

That is, $P_{\Lambda,s}(x, y; d\omega)$ is the convex combination of the Wiener measures corresponding to Neumann and Dirichlet boundary conditions. We denote that

$$\Xi_\Lambda(s) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\pi \in S_n} \int_{\Lambda^n} d(x)_n \int_{\Omega^n} P_{\Lambda,s}((x)_n, \pi(x)_n; d(\omega)_n)\, e^{-U((\omega)_n)} \tag{5.10}$$

and for any function $A((x)_n)$ invariant under S_n

$$\rho_{\Lambda,s}(A) = \Xi_\Lambda(s)^{-1} \sum_{n=0} \frac{z^n}{n!} \sum_{\pi \in S_n} \int_{\Lambda^n} d(x)_n\, A((x)_n) \int_{\Omega^n} P_{\Lambda,s}((x)_n, \pi(x)_n; d(\omega)_n) \times e^{-U((\omega)_n)} \tag{5.11}$$

From the definitions it follows that

$$\begin{aligned} \Xi_\Lambda(1) &= \Xi_{0,\Lambda}, \qquad \Xi_\Lambda(0) = \Xi_{\infty,\Lambda} = \Xi_\Lambda \\ \rho_{\Lambda,1}(A) &= \rho_{0,\Lambda}(A) \qquad \text{and} \qquad \rho_{\Lambda,0}(A) = \rho_\Lambda(A) \end{aligned} \tag{5.12}$$

Let $\mathbf{K}_{\Delta,l,s}$ and $\bar{\mathbf{K}}_{\Delta,l,s}$ be the operators corresponding to the kernels

$$\mathbf{K}_{\Delta,l,s}(x, y) = \int_{\mathscr{E}(\Delta,l)} P_{\Lambda,s}(x, y; d\omega) \tag{5.13}$$

and

$$\begin{aligned} \bar{\mathbf{K}}_{\Delta,l,s}(x, y) &= P_{\Lambda(\Delta,l+1)}(sE^{(\beta/4)\Delta_{0,\Lambda}} + (1-s)\, e^{(\beta/4)\Delta_\Lambda})\, P_{\Lambda(\Delta,l+1)}(x, y) \\ &= \int_\Omega \chi_{\Lambda(\Delta,l+1)}(\omega)\, P^{\beta/2}_{\Lambda,s}(x, y; d\omega) \end{aligned} \tag{5.14}$$

respectively [see (3.2.23) and (3.2.24)]. Following the method used in the proof of Proposition III.2.2 of Ref. 10 it is easy to check that Lemma 3.2.3 holds for $\mathbf{K}_{\Delta,l,s}$ and $\bar{\mathbf{K}}_{\Delta,l,s}$. Furthermore all arguments and results in Section 4 hold even if one replaces $P^\beta_\Lambda(x, y; d\omega)$ by $P^\beta_{\Lambda,s}(x, y; d\omega)$.

We next use the fundamental theorem of calculus to get

$$\begin{aligned} P_{0,\Lambda} - P_\Lambda &= \frac{1}{|\Lambda|} \int_0^1 d(\log \Xi_\Lambda(s)) \\ &= \frac{1}{|\Lambda|} \int_0^1 \left[\frac{1}{\Xi_\Lambda(s)}\right] \frac{d}{ds} (\Xi_\Lambda(s))\, ds \end{aligned} \tag{5.15}$$

From (5.10) it follows that

$$\frac{d}{ds}\Xi_\Lambda(s) = \sum_{Q(r):Q(r)\cap\Lambda\neq 0} \Xi'_\Lambda(s, r) \tag{5.16}$$

where

$$\begin{aligned}\Xi'_\Lambda(s, r) = \sum_{n=0}^{\infty} \frac{z^n}{(n-1)!} \sum_{\pi\in S_n} \int_{Q(r)\cap\Lambda} dx_1 \int_{\Lambda^{n-1}} d(x)_{n-1}\\ \times \int_\Omega [P_{\Lambda,s}(x_1, \pi(x_1); d\omega_1)|_0^1]\\ \times \int_{\Omega^{n-1}} P_{\Lambda,s}((x)_{n-1}, \pi(x)_{n-1}; d(\omega)_{n-1})\\ \times e^{-U((\omega)_n)}\end{aligned} \tag{5.17}$$

We will show that

$$\Xi'_\Lambda(s, r) \leqslant \left\{e^D \exp\left[-\frac{1}{90\beta} d(Q(r), \partial\Lambda)^2\right]\right\} \Xi_\Lambda(s) \tag{5.18}$$

for some constant $D>0$, where $d(Q(r), \partial\Lambda)$ is the distance between $Q(r)$ and $\partial\Lambda$. Then from (5.15), (5.16), and (5.18) it follows that

$$\begin{aligned}|P_{0,\Lambda} - P_\Lambda| &\leqslant c|\partial\Lambda|/|\Lambda|\\ &\to 0 \quad \text{as} \quad \Lambda\to\mathbf{R}^\nu\end{aligned}$$

In the remainder of this section we prove the bounds in (5.18).

Let $\mathbf{K}'_{\Delta,l}$ be the operators of which the kernels are given by

$$\mathbf{K}'_{\Delta,l}(x, y) = \int_{\mathscr{E}(\Delta,l)} [P^\beta_{\Lambda,s}(x, y; d\omega)\,|_0^1 \tag{5.19}$$

We then have the following result:

Lemma 5.3. For $l = -1, 0, 1, 2,...$ there are constants $c_1>0$ and $c_2>0$ such that

(a) $\mathrm{Tr}_{L^2(\Lambda)}(\mathbf{K}_{\Delta,l,s}) \leqslant c_1\,|A(\Delta, l+1)|\,\exp\left(-\frac{1}{16\beta} l^2\right)$

(b) $|\mathrm{Tr}_{L^2(\Lambda)}(\mathbf{K}'_{\Delta,l})| \leqslant c_2 \exp\left(\frac{-1}{60\beta} l^2\right) \exp\left[-\frac{1}{60\beta} d(\Delta, \partial\Lambda)^2\right]$

Proof. As we stated, the part (a) follows from the method similar to that used in the proof of Lemma 3.2.3 (see the proof of Proposition III.2.2 of Ref. 10). To prove the part (b) we note that

$$\mathbf{K}'_{\Delta,l}(x,x)=\int_{\mathscr{E}(\Delta,l)}[P_{0,\Lambda}(x,x;d\omega)-P_\Lambda(x,x;d\omega)]$$

If $l\leqslant\frac{2}{3}d(\Delta,\partial\Lambda)$, then

$$\begin{aligned}|\mathbf{K}'_{\Delta,l}(x,x)|&\leqslant\left|\chi_{\Lambda(\Delta,l+1)}(x)\int[P_{0,\Lambda}(x,x;d\omega)-P_\Lambda(x,x;d\omega)]\right|\\&\leqslant\left|\chi_{\Lambda(\Delta,l+1)}(x)\left[\int P(x,x;d\omega)-\int P_{0,\Lambda}(x,x;d\omega)\right]\right|\\&\quad+\left|\chi_{\Lambda(\Delta,l+1)}(x)\left[\int P(x,x;d\omega)-\int P_\Lambda(x,x;d\omega)\right]\right|\\&\leqslant c\exp\left\{-\frac{1}{4\beta}\left[\frac{1}{3}d(\Delta,\partial\lambda)\right]^2\right\}\chi_{\Lambda(\Delta,l+1)}(x)\end{aligned}$$

Here we have use Theorem 6.3.8 of Ref. 1 to get the last inequality. Hence

$$|\mathrm{Tr}(\mathbf{K}'_{\Delta,l})|\leqslant c\exp\left[-\frac{1}{37\beta}d(\Delta,\partial\Lambda)^2\right]$$

On the other hand, if $l>\frac{2}{3}d(\Delta,\partial\Lambda)$, then

$$\begin{aligned}|\mathrm{Tr}(\mathbf{K}'_{\Delta,l})|&\leqslant\int d^\nu x\int_{\mathscr{E}(\Delta,l)}P_{0,\Lambda}(x,x;d\omega)\\&\quad+\int d^\nu x\int_{\mathscr{E}(\Delta,l)}P_\Lambda(x,x;d\omega)\\&\leqslant c\,|\Lambda(\Delta,l+1)|\exp\left(-\frac{1}{16\beta}l^2\right)\\&\leqslant c'\exp\left[-\frac{1}{37\beta}d(\Delta,\partial\Lambda)^2\right]\end{aligned}\tag{5.20}$$

Here we have use the part (a) of the lemma to get the second inequality. Thus we conclude that

$$|\mathrm{Tr}(\mathbf{K}'_{\Delta,l})|\leqslant c\exp\left[-\frac{1}{37\beta}d(\Delta,\partial\Lambda)^2\right]\tag{5.21}$$

By the method used in (5.20) we also have

$$|\mathrm{Tr}(\mathbf{K}'_{\Delta,l})| \leqslant c \exp\left[-\frac{1}{18\beta} l^2\right] \tag{5.22}$$

Combining (5.21) and (5.22) we prove the part (b). ▮

We now begin to prove Proposition 5.2.

Proof of Proposition 5.2. As we remarked before it is sufficient to show the bounds in (5.18). The basic idea is a modification of the method used in Section 3.2. Since the detailed derivation of the bounds in (5.18) would be very long and since the main idea is same as that in Section 3.2, we will give a sketch of the proof by giving necessary modifications and replacements. We leave the details to the reader.

For a given $Q(r)$, $Q(r) \cap \Lambda \neq \phi$, we use the following abused notation:

$$\Lambda_q = \prod_{i=1}^{\nu} \left[-l_q - \frac{1}{2} + r^i, l_q + \frac{1}{2} + r^i\right] \tag{5.23}$$

where l_q is given in (3.1.9). Λ_q is the closed cube centered at r. In the rest of this section Λ_q is the cube defined in (5.23), not in (3.1.9). For a given integer we use the decompositions in (3.1.11) and (3.1.19) for Ω^{n-1}:

$$\Omega^{n-1} = \mathscr{E}_0 \cup \left(\bigcup_{q \geqslant q_0} \bigcup_{k} \mathscr{E}_{q,k-1}\right)$$

Following each step used to derive the expressions in (3.1.20) and (3.1.26) we obtain

$$\Xi'_\Lambda(s, r) = \tilde{G}_0 + \sum_{n=0}^{\infty} \sum_{q \geqslant q_0} \sum_{k} \tilde{G}^{(n)}(q, k) \tag{5.24}$$

where $\tilde{G}_0$ is the expression obtained by repacing Ω^{n-1} in (5.17) by $\mathscr{E}_0$ and

$$\begin{aligned}\tilde{G}^{(n)}(q, k) = {} & \frac{z^n}{(k-1)!\,(n-k)!} \sum_{\pi \in S_n} \int_{Q(r) \cap \Lambda} dx_1 \int_{\Lambda^{n-1}} d(x)_{n-1} \\ & \times \int_{\Omega} [P_{\Lambda,s}(x_1, \pi(x_1); d\omega_1) |_0^1] \\ & \times \int_{\bar{\mathscr{E}}_{q,k-1}} P_{\Lambda,s}((x)_{n-1}, \pi(x)_{n-1}; d(\omega)_{n-1})\, e^{-U((\omega)_n)}\end{aligned} \tag{5.25}$$

where $\bar{\mathscr{E}}_{q,k-1} = \{(\omega)_{n-1} \in \mathscr{E}_q : \omega_i(\tau) \in \Lambda_{q+1}$ for some $\tau \in [0, \beta]$, $i = 2,..., k\}$.

We assert that for q_0 sufficiently large one can choose $1 < l < p < \gamma(\nu, 2)$ such that the following bounds hold: For any $b > 0$ there exist positive constants c_1, c_2 and ε such that

$$U(\omega_1) + bV(\omega_1) + W(\omega_1, (\omega)_{n-1}) \geqslant c_1 \tag{5.26}$$

on $\mathscr{E}_0$, and

$$\begin{aligned} &U((\omega)_k) + bV((\omega)_k) + W((\omega)_k, (\omega)_{n-k}) \\ &\qquad \geqslant c_3\{|\Lambda_q|^{1+[(l-1)(1+\varepsilon)/p]} + k^{1+\varepsilon}\} \end{aligned} \tag{5.27}$$

on $\bar{\mathscr{E}}_{q,k-1}$. We prove our assertion. We note that $E_q < |\Lambda_q|^l$ on $\mathscr{E}_0$ for all $q \geqslant q_0$. In order to show (5.26) we consider two cases: If $\omega_1(\tau) \in \Lambda_{q_0}$ for all $\tau \in [0, \beta]$, then (5.26) follows from the lower regularity condition [Assumption A(b)]. If $\omega_1(\tau_0) \notin \Lambda_{q_0}$ for some $\tau_0 \in [0, \beta]$, then (5.26) follows from the methods similar to those used in the proofs of Lemma 4.7 and (4.15). Next, the bound in (5.27) can be obtained by the method used in the proof of Theorem 3.1.1 (see Section 4). We leave the detailed proofs of (5.26) and (5.27) to the reader.

As in (3.2.9) we define

$$\begin{aligned} \tilde{F}_0^{(n)}(Q(r), l) &= \sum_{\pi \in S_n} \int_{Q(r)} dx_1 \int_{\Lambda^{n-1}} d(x)_{n-1} \tilde{\mathbf{K}}_0((x)_n, \pi(x)_n) \\ &\quad \times \tilde{F}^{(n)}(q, k; (Q(r), l_1), (\Delta_2, l_2), ..., (\Delta_k, l_k)) \\ &= \sum_{\pi \in S_n} \int_{Q(r)} dx_1 \int_{\Lambda^{n-1}} d(x)_{n-1} \tilde{\mathbf{K}}((x)_n, \pi(x)_n) \end{aligned} \tag{5.28}$$

where

$$\begin{aligned} \tilde{\mathbf{K}}_0((x)_n, \pi(x)_n) &= \int_{\mathscr{E}(Q(r),l)} [P_{\Lambda,s}(x_1, \pi(x)_1, d\omega_1)|_0^1] \\ &\quad \times \int_{\bar{\mathscr{E}}_{q,k-1}} P_{\Lambda,s}((x)_{n-1}, \pi(x)_{n-1}; d(\omega)_{n-1})\, e^{-U((\omega)_{n-1})} \\ \tilde{\mathbf{K}}((x)_n, \pi(x)_n) &= \int_{\mathscr{E}(Q(r),l_1)} [P_{\Lambda,s}(x_1, \pi(x)_1; d\omega_1)\, |_0^1] \\ &\quad \times \int_{\bar{\mathscr{E}}(q,k-1;(\Delta_2,l_2)...,(\Delta_k,l_k))} P_{\Lambda,s}((x)_{n-1}, \pi(x)_{n-1}; d(\omega)_{n-1}) \\ &\quad \times e^{-U((\omega)_{n-k})} \end{aligned} \tag{5.29}$$

Let $\mathbf{E}_{n,s}$ be the operator of which its kernel is given by

$$\mathbf{E}_{n,s}((x)_n, (y)_n) = \int_{\Omega^n} P_{\Lambda,s}((x)_n, (y)_n; d(\omega)_n)\, e^{-U((\omega)_n)} \tag{5.30}$$

Notice that

$$\Xi_\Lambda(s) = \sum_{n=0}^{\infty} \mathrm{Tr}_{\mathscr{H}_n^{(s)}(\Lambda)}(\mathbf{E}_{n,s}) \tag{5.31}$$

Using the bounds in (5.26) and (5.27) and following the steps to derive (3.2.8) we obtain the following bounds:

$$\tilde{G}_0 \leqslant c_1 \sum_{n=0}^{\infty} \frac{z^n}{(n-1)!} \sum_{l=-1}^{\infty} \exp[4vb(l+2)^2]\, \tilde{F}_0(Q(r), l)$$

$$\begin{aligned}\tilde{G}^{(n)}(q,k) \leqslant c_2 \Bigg\{ &\frac{z^n}{(k-1)!\,(n-k)!} \sum_{\Delta_2 \subset \Lambda_{q+1}} \cdots \sum_{\Delta_k \subset \Lambda_{q+1}} \sum_{l_1=-1}^{\infty} \cdots \sum_{l_k=-1}^{\infty} \\ &\times \exp\left[4vb \sum_{i=1}^{k} (l_i+2)^2\right] \tilde{F}^{(n)}(q,k;(Q(r),l_1),\dots,(\Delta_k,l_k))\Bigg\} \\ &\times \exp[-c(|\Lambda_q|^{1+[(l-1)(1+\varepsilon)/p]} + k^{1+\varepsilon})]\end{aligned} \tag{5.32}$$

What we have done up to now is the derivation of the forms in (5.24) and (5.32) so that the methods used in Section 3.2 can be applied. To prove the bound in (5.18) we use the following replacements in Section 3.2:

$$\left.\begin{array}{l} \mathbf{K}_{\Delta_1,l_1}(x_1, y_1) \\ \mathbf{K}_{\Delta_i,l_i}(x_i, y_i) \\ i=2,\dots,k \\ \exp[-\beta H_\Lambda^{(n)}] \end{array}\right\} \text{ by } \left\{\begin{array}{l} \mathbf{K}'_{Q(r),l_1}(x_1, y_1) \\ \mathbf{K}_{\Delta_i,l_i,s}(x_i, y_i) \\ i=2,\dots,k \\ \mathbf{E}_{n,s} \end{array}\right.$$

We then use Lemma 5.3 instead of Lemma 3.2.3. Following each step used in the second part of (3.2.9) in Section 3.2, one may get the bounds of the following forms from (5.32) and (5.33):

$$\tilde{G}_0 \leqslant c_1 \exp\left[-\frac{1}{90\beta} d(Q(r), \partial\Lambda)^2\right] \sum_{n=1}^{\infty} \mathrm{Tr}_{\mathscr{H}_n^{(s)}(\Lambda)}(\mathbf{E}_{n,s})$$

$$G^{(n)}(q,k) \leqslant c_2 \exp[-c(q+k)^{1+\varepsilon}] \exp\left[\frac{-1}{90\beta} d(Q(r), \partial\Lambda)^2\right] \times \mathrm{Tr}_{\mathscr{H}_{n-k}^{(s)}(\Lambda)}(\mathbf{E}_{n-k,s})$$

The bound in (5.18) follows from (5.24), (5.31) and the above inequalities. This completes the proof of Proposition 5.2. ▮

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Appendix. Corrections of Printing Errata in Ref. 10

$n\downarrow$ means nth line from the top; $n\uparrow$ means nth line from the bottom:

Page		Replace	With
11	$7\downarrow$	$\exp\left(-\frac{l}{16\beta}\right)$	$\exp\left(-\frac{l^2}{16\beta}\right)$
25	$14\downarrow$	$-\beta H_\Lambda^{(n')}$	$\exp[-\beta H_\Lambda^{(n')}]$
29	$12\downarrow$	$\sum_{\substack{r \in \Lambda_{4q}^c \\ r \in \Lambda_{2q}^c}}$	$\sum_{\substack{r \in \Lambda \\ r' \in \Lambda_{2q}^c}}$
29	$8\uparrow$	$\bar{n}(r', \tau)\, n(r, \tau)\, n(r, \tau)$	$\bar{n}(r', \tau)\, n(r, \tau)$
30	$15\downarrow$	$\sum_{r \in \Lambda_{4q}^c}$	$\sum_{r \in \Lambda}$
32	$4\downarrow$	$\int_d^v (x)$	$\int d^\nu x$

REFERENCES

1. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vols. I and II (Springer, New York, 1981).
2. R. Esposito, F. Nicolo, and M. Pulvirenti, Superstable interactions in quantum statistical mechanics: Maxwell–Boltzmann statistics, *Ann. Henri Poincaré*, **XXXVI**(2):127–158 (1982).
3. D. C. Brydges and P. Federbush, The cluster expansion in statistical mechanics, *Commun. Math. Phys.* **49**:233 (1976); The cluster expansion for potentials with exponential fall-off, *Commun. Math. Phys.* **53**:19 (1977).
4. M. Duneau and B. Souillard, Existence of Green's functions for dilute Bose gases, *Commun. Math. Phys.* **31**:113 (1973).

5. J. Fröhlich and Y. M. Park, Correlation inaqualities and the thermodynamic limit for classical and quantum continuous systems, II. Bose–Einstein and Fermi–Dirac statistics, *J. Stat. Phys.* **23**:701 (1980).
6. J. Ginibre, Some application of functional integration in statistical mechanics, in *Statistical Mechanics and Field Theory*, C. de Witt and R. Stora, eds. (Gordon and Breach, New York, 1971).
7. J. Glimm and A. Jaffe, *Quantum Physics; A Functional Integral Point of View* (Springer, New York, 1982).
8. J. L. Lebowitz and E. Presutti, Statistical mechanics of systems of unbounded spins, *Commun. Math. Phys.* **50**:195 (1976).
9. E. H. Lieb and J. L. Lebowitz, *Adv. Math.* **9**:316 (1972).
10. Y. M. Park, Bounds on exponentials of local number operators in quantum statistical mechanics, *Commun. Math. Phys.* **94**:1–33 (1984).
11. Y. M. Park, Quantum statistical mechanics of unbounded continuous spin systems, preprint.
12. M. Reed and B. Simon, *Method of Modern Mathematical Physics*, Vol. II: *Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975).
13. D. W. Ruelle, *Statistical Mechanics* (Benjamin, *New York*, 1969).
14. D. W. Ruelle, Superstable interactions in classical statistical mechanics, *Commun. Math. Phys.* **18**:127 (1970).
15. D. W. Ruelle, Probability estimates for continuous spin systems, *Commun. Math. Phys.* **50**:189 (1976).
16. D. W. Ruelle, Analyticity of Green's functions for dilute quantum gases, *J. Math. Phys.* **12**:901 (1971).